AN

# ELEMENTARY TREATISE

ON

# MENSURATION

AND

# PRACTICAL GEOMETRY,

TOGETHER WITH

## NUMEROUS PROBLEMS OF PRACTICAL IMPORTANCE

IN

# MECHANICS.

BY

WILLIAM VOGDES, LL.D.

PROFESSOR OF MATHEMATICS IN THE CENTRAL HIGH SCHOOL OF PHILADELPHIA

AUTHOR OF THE UNITED STATES ARITHMETIC.

[PART FIRST.]

PHILADELPHIA:

E. C. & J. BIDDLE & CO., No. 508 MINOR ST.

*(Between Market and Chestnut, and Fifth and Sixth Sts.)*

1861.

Geo. Bland.
1863.

George Bland

Roxborough

Pennsylvan

April, 1863.

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*Chamber of the Controllers of Public Schools,*
*First School District of Pennsylvania.*

PHILADELPHIA, November 15th, 1849.

AT a meeting of the Controllers of Public Schools, First School District of Pennsylvania, held at the Controllers' Chamber, on Tuesday, November 13th, 1849, the following Resolution was adopted:—

*Resolved*, That Vogdes' Mensuration be introduced as a Class Book into the Grammar Schools of the District.

From the Minutes.

ROBERT J. HEMPHILL, *Secretary.*



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# PREFACE.

It has been the design of the author, in the following pages, to compile a work adapted, by its practical character, to the wants of those of the rising generation, who, not being able to command a collegiate education, are fitting themselves to fill useful stations in society as mechanics, merchants, &c. With this end in view, it was deemed inexpedient to devote any portion of the book to *theoretical* demonstrations of the principles involved in the rules given, when those demonstrations are based upon principles illustrated by more advanced branches of mathematical science, with which the pupil is supposed to be unacquainted. By pursuing this course, room was afforded for the introduction of more numerous examples illustrating the respective rules, than could otherwise have been given without the enlargement of the work beyond expedient limits. The introduction of these examples, it is believed, will enhance the value of the work in the estimation of teachers generally, inasmuch as the operation of the rules is more likely to be permanently impressed on the mind of the pupil, by long continued practice, than by the solution of one or two problems only.

In a treatise on Mensuration, little that is *new* can be looked for, other than the collection and judicious arrangement of matters not heretofore presented to the public in a form adapted to the purposes for which this

work is designed. The greater portion of this volume has been derived from the works of Bonnycastle, Haswell, Hutton, Gregory, and Grier. To some of these, special acknowledgment of the obligation has been made in the subsequent pages.

The application of science to the arts of industry has been one of the most potent operative causes of the rapid increase of our country in wealth and power. In the hope that this compilation may be found a useful assistant to the teacher who is engaged in preparing pupils for an active participation in these industrial pursuits, the author submits it to the inspection of his colabourers in this field.

*Philadelphia, June* 27, 1846.

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A KEY to this work has been published.

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## Table I.—Square Measure.

| Square Inches. | Square Links. | Square Feet. | Square Yards. | Square Poles, Perches, or Rods. | Square Chains. | Roods. | Acres. | Square Mile. |
|---|---|---|---|---|---|---|---|---|
| $62\frac{454}{625}$ | = 1 | | | | | | | |
| 144 | $2\frac{322}{1089}$ | = 1 | | | | | | |
| 1296 | $20\frac{80}{121}$ | 9 | = 1 | | | | | |
| 39204 | 625 | $272\frac{1}{4}$ | $30\frac{1}{4}$ | = 1 | | | | |
| 627264 | 10000 | 4356 | 484 | 16 | = 1 | | | |
| 1568160 | 25000 | 10890 | 1210 | 40 | $2\frac{1}{2}$ | = 1 | | |
| 6272640 | 100000 | 43560 | 4840 | 160 | 10 | 4 | = 1 | |
| 4014489600 | 64000000 | 27878400 | 3097600 | 102400 | 6400 | 2560 | 640 | = 1 |

## Table II.—Lineal Measure.

| Inches. | | Gunter's Link. | | Feet. | | Yards. | | Fathoms. | | Rods, Poles, or Perches. | | Gunter's Chains. | | Furlongs. | | Mile. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $7\frac{23}{25}$ | = | 1 | | | | | | | | | | | | | | |
| 12 | | $1\frac{17}{33}$ | = | 1 | | | | | | | | | | | | |
| 36 | | $4\frac{6}{11}$ | | 3 | = | 1 | | | | | | | | | | |
| 72 | | $9\frac{1}{11}$ | | 6 | | 2 | = | 1 | | | | | | | | |
| 198 | | 25 | | $16\frac{1}{2}$ | | $5\frac{1}{2}$ | | $2\frac{3}{4}$ | = | 1 | | | | | | |
| 792 | | 100 | | 66 | | 22 | | 11 | | 4 | = | 1 | | | | |
| 7920 | | 1000 | | 660 | | 220 | | 110 | | 40 | | 10 | = | 1 | | |
| 63360 | | 8000 | | 5280 | | 1760 | | 880 | | 320 | | 80 | | 8 | = | 1 |

## Table III.—Cubic or Solid Measure.

| Cubic Inches. | | Cubic Feet. | | Cubic Yards. | | Cub. Poles, Rods, or Perches. | | Cub. Furlongs. | | Cub. Mile. |
|---|---|---|---|---|---|---|---|---|---|---|
| 1728 | = | 1 | | | | | | | | |
| 46656 | | 27 | = | 1 | | | | | | |
| 7762392 | | $4482\frac{1}{8}$ | | $166\frac{3}{8}$ | = | 1 | | | | |
| 496793088000 | | 287496000 | | 10648000 | | 64000 | = | 1 | | |
| 2544358061056000 | | 147197952000 | | 5451776000 | | 32768000 | | 512 | = | 1 |

## Table IV.—Other Measures.

40 cubic feet of round timber make - - 1 ton, T.
50 cubic feet of hewn timber " - - - 1 ton, T.
40 cubic feet - - - " - 1 ton of shipping.
128 cubic feet, or 8 feet in length, and 4 in breadth, and 4 in height, make } 1 cord of wood.
$24\frac{3}{4}$ cubic feet, or $16\frac{1}{2}$ feet in length, $1\frac{1}{2}$ in breadth and 1 in height make } 1 perch of stone.
282 cubic inches - - - " 1 gallon, ale measure.
231 cubic inches - - - " 1 gallon, wine measure.
277.274 cubic inches (Eng.) " 1 imperial gallon.
$268\frac{4}{5}$ cubic inches - - " 1 gallon, dry measure.
2150.42 cubic inches - - " - - 1 bushel.

# EXPLANATION OF THE CHARACTERS USED IN THIS WORK.

$+$ denotes *plus*, or *more*. The sign of addition, signifying that the numbers between which it is placed are to be added together. Thus, $8 + 5$, denotes that 5 is to be added to 8. Geometrical lines are generally represented by capital letters. Thus, A B + C D, signifies that the line A B is to be added to the line C D.

$-$ denotes *minus*, or *less*. The sign of subtraction, signifying that the latter of the two numbers between which it is placed is to be taken from the former. Thus, $4 - 2$, denotes that 2 is to be taken from 4. In geometrical lines, also, A B — C D, signifies that the line C D is to be subtracted from the line A B.

$\times$ denotes *into*, or *by*. The sign of multiplication, signifying that the numbers between which it is placed are to be multiplied together. Thus, $7 \times 5$, denotes that 7 is to be multiplied by 5. In geometrical lines, also, A B $\times$ C D, signifies that the number of units in the line A B is to be multiplied by the number of units in the line C D. Instead of the sign $\times$, a point is sometimes employed. Thus, A B . C D, is the same as A B $\times$ C D.

$\div$ denotes *divided by*. The sign of division, signifying that the former of the two numbers between which it is placed is to be divided by the latter. Thus, $6 \div 3$, denotes that 6 is to be divided by 3. This is also expressed by placing the dividend above a line and the divisor below it. Thus, $\frac{6}{3}$ denotes that 6 is to be divided by 3. In geometrical lines, also, A B $\div$ C D, signifies that the line A B is to be divided by the line C D, or thus, $\frac{A\,B}{C\,D}$.

: *is to*
:: *so is*
: *to*

$\left.\right\}$ are proportionals, signifying that the numbers between which they are placed are proportional. Thus, as 2 : 4 :: 8 : 16, denotes that the number 2 has the same proportion to 4 that 8 has to 16.

$=$ denotes *equal to*. The sign of equality, signifying that the numbers between which it is placed are equal to each other, Thus, 2 poles $+$ 2 poles $=$ 4 poles $=$ 22 yards $=$ 1 chain $=$ 100 links.

( ) the parenthesis; [ ] the crotchet; and { } the brace, are signs made use of to connect two or more quantities together, and they are synonymous with regard to their application; for

$$[(7+4)-5]\times 8=(11-5)\times 8=6\times 8=48,$$

is the same as

$$\{(7+4)-5\}\times 8=(11-5)\times 8=6\times 8=48.$$

The parenthesis which includes the 7 and 4, serves as a chain to link them together, and shows that they are to be added together before the number 5 is subtracted; and the brace also shows that the numerals which it includes must be operated upon; and the result multiplied by the number 8.

$^2$ This sign is placed above a quantity, signifying that the quantity is to be squared.

Thus, $(5+2)^2=7^2=7\times 7=49$.

$^3$ This sign is placed above a quantity, signifying that the quantity is to be cubed.

Thus, $[(7+9)-8]^3=(16-8)^3=8^3=8\times 8\times 8=512$.

$\surd$ is a radical sign, signifying that the quantity before which it is placed is to have the square root extracted. Thus,

$$\sqrt{[(8+6)-(4\times 2)]^2+\{[(12\times 9)\div 3]-[8+(9-2)\}+\{[(7+5)\times 12]\div[3\times(4\div 2)]\}}=$$
$$\surd\{(14-8)^2+[(108\div 3)-(8+7)]+[(12\times 12)\div(3\times 2)]\}$$
$$=\surd[6^2+(36-15)+(144\div 6)]=\surd(36+21+24)$$
$$=\surd 81=9.$$

$\sqrt[3]{}$ is a radical sign, signifying that the quantity before which it is placed is to have the cube root extracted.

Thus, $\sqrt[3]{[(6\times 4\times 3)-8]}=\sqrt[3]{(72-8)}=\sqrt[3]{64}=4$.

$\therefore$ denotes *therefore*.

$\perp$ denotes a perpendicular.

$<$ denotes an angle.

$\triangle$ denotes a triangle.

# PRACTICAL GEOMETRY.

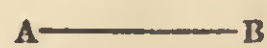
### DEFINITIONS.

§ 1. Practical Geometry is a mechanical method of describing mathematical figures, by means of the scale and compasses, or other instruments proper for the purpose. It is founded upon the properties and relations of certain magnitudes, which may be found treated at large in the works of Euclid and other authors. The definitions of the principal figures are as follows:

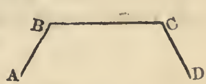
1. A point, considered mathematically, is that which has no parts or dimensions, but merely position.

2. A line is length without breadth, and its bounds or extremes are points.

3. A right or straight line is that which lies evenly between its extreme points; as A B.

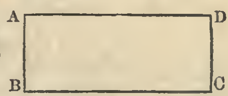



4. A broken line is one which changes its direction at intervals so large that they can be perceived; as A B C D.



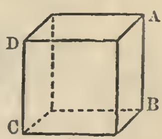

5. A curved line is one which changes its direction at intervals so small that they cannot be perceived; as A B.


6. A superficies is that which has length and breadth only; and its bounds or extremes are lines; as A B C D.




7. A plane, or plane superficies, is that which is everywhere perfectly flat and even. Or, in other words, it is that with which a right line will every way coincide.

8. A body, or solid, is that which has length, breadth, and thickness, and its bounds, or extremes, are superficies; as A B C D.

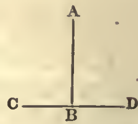


9. A plane rectilineal angle is the inclination or opening of two right lines, which meet in a point without cutting each other; as A B C.


Here it is to be observed that the greater or less length of the lines makes no alteration in the angle.

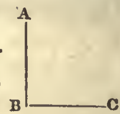
10. One right line is said to be perpendicular to another, when the angles on each side of it are equal.

Thus A B is perpendicular to C D. Angles are of three kinds; being either right, acute, or obtuse.

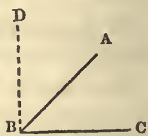



11. A right angle is that which is formed by two right lines, that are perpendicular to each other; as A B C.

Any angle differing from a right angle, whether it be greater or less, is called an oblique angle, and the lines that form it are called oblique lines.



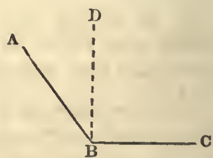

12. An acute angle is that which is less than a right angle; as A B C.



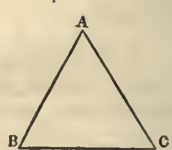

13. An obtuse angle is that which is greater than a right angle, as A B C.

14. A figure is a space bounded by one or more lines.

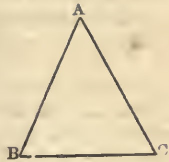
15. All plane figures bounded by three right lines are called triangles, and receive different denominations according to the nature of their sides and angles.



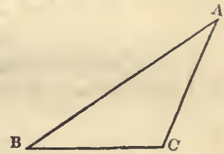

16. An equilateral triangle is that which has all its sides equal; as A B C.



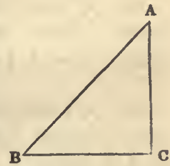
17. An isosceles triangle is that which has only two of its sides equal; as A B C.




18. A scalene triangle is that which has all its three sides unequal; as A B C.

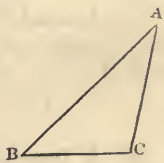



19. A right-angled triangle is that which has one right angle; the side opposite to the right angle is called the hypothenuse, and the other two sides the legs; as A B C, where A B is the hypothenuse, and B C, A C the two legs, or base and perpendicular.

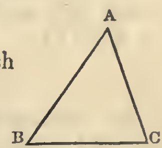



Any triangle differing from a right-angled one is frequently called an oblique-angled triangle.

20. An obtuse-angled triangle is that which has one obtuse angle; as A C B, where C is the obtuse angle.

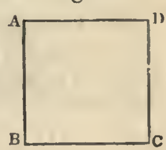



21. An acute-angled triangle is that which has all its angles acute; as A B C.



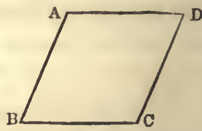

22. All plane figures bounded by four right lines, are called quadrangles, or quadrilaterals; and receive different names according to the nature of their sides and angles.

23. A square is a quadrilateral, whose sides are all equal, and its angles all right angles; as A B C D.



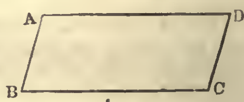
A square is also an instrument used by artificers for what is called squaring their work; being of various forms, as the T square, normal square, &c.

24. A rhombus is a quadrilateral whose sides are all equal, but its angles not right angles; as A B C D.



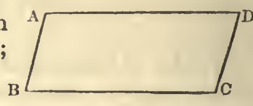

This figure, by mechanics, is generally called a lozenge; and both it and the square belong to the class of parallelograms.

25. A parallelogram is a quadrilateral whose opposite sides are parallel; as A B C D.



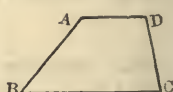

26. A rectangle is a parallelogram whose angles are all right angles; as A B C D.


27. A rhomboid is a parallelogram whose angles are not right angles; as A B C D.

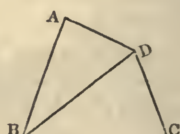



28. A trapezium is a quadrilateral which hath not its opposite sides parallel; as A B C D.


29. A trapezoid is a quadrilateral, having two of its opposite sides parallel; as A B C D.




30. The right line joining any two opposite angles of a quadrangle, or quadrilateral, is called its diagonal; as B D in the figure A B C D.



31. All plane figures contained under more than four sides are called polygons; and receive different names, according to the number of their sides, or angles.

32. Thus, polygons having five sides are called pentagons; those of six sides, hexagons, those of seven, heptagons; and so on.

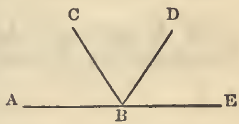
33. A regular polygon is that which has all its sides as well as its angles equal to each other, and if the sides or angles are unequal, it is called an irregular polygon.

An equilateral triangle is also a regular polygon of three sides, and a square is one of four sides.

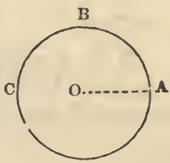
34. Parallel right lines are such as are everywhere at an equal distance from each other; or which, if infinitely produced, would never meet; thus A B is parallel to C D.


35. The base of any figure is that side on which it is supposed to stand; and its altitude is the perpendicular falling upon the base from the opposite angle.

36. An angle is usually denoted by three letters, the one which stands at the angular point being always to be read in the middle, as A B C, C B D, D B E, &c.




37. A circle is a plane figure bounded by a curve line called the circumference or periphery, which is everywhere equidistant from a point within, called its centre; and is formed by the revolution of a right line (O A) about one of its extremities (O), which remains fixed.

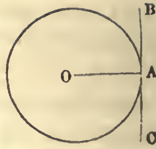



38. The centre of a circle is the point (O) about which it is described; and the circumference or periphery is the line or boundary A B C A, by which it is contained.

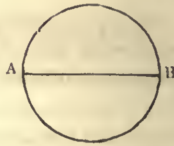
The circumference itself, as well as the space which is bounded by it, is also, for the sake of conciseness, sometimes called a circle.

39. The radius of a circle is a right line drawn from the centre to the circumference; as O A.

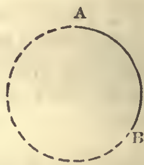
A tangent is a line touching a circle, and which produced, does not cut it, as B A C.



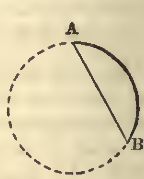

40. The diameter of a circle is a right line passing through the centre, and terminated both ways by the circumference; as A B.



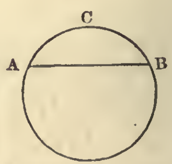

41. An arc of a circle is any part of its circumference, or periphery; as A B.



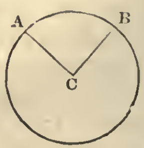

42. A chord is a right line which joins the extremities of an arc; as A B.



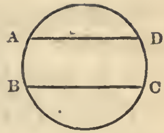

43. A segment of a circle is the space contained between an arc and its chord; as A B C.



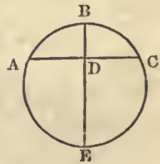

44. A sector is the space contained between an arc and the two radii drawn to its extremities; as A B C.



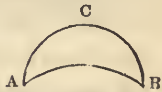
45. A zone is a part of a circle included between two parallel chords and their intercepted arcs; as A B C D.




46. The versed sine or height of an arc, is that part of the diameter contained between the middle of the chord and the arc; as D B, or D E.



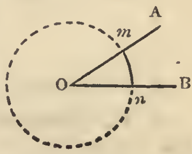

47. A lune is the space included between the intersecting arcs of two eccentric circles; as A B C.




48. A semicircle is a half of a circle; a quadrant is a quarter of a circle; a sextant the sixth part of it, and an octant the eighth part, where it may be observed that these names are often applied to instruments used for taking angles.

49. The circumference of every circle is supposed to be divided into 360 equal parts called degrees; each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds.

50. The measure of any right-lined angle is an arc of a circle contained between the two lines which form that angle, the angular point being the centre; thus the angle A O B is measured by the arc $m\,n$.




The angle is estimated by the number of degrees, minutes, &c. contained in the arc; whence a right angle is an angle of 90 degrees or $\frac{1}{4}$ of the circumference.

## AXIOMS.

1. An axiom is an established principle, or self-evident truth, requiring no other conviction than that which arises from a proper understanding of the terms in which it is proposed.

2. Things which are equal to the same thing are equal to each other.

3. If equals be added to equals the wholes will be equal.

4. If equals be taken from equals the remainders will be equal.

5. If equals be added to unequals the wholes will be unequal.

6. If equals be taken from unequals the remainders will be unequal.

7. Things which are half, double, or any number of times the same thing, are equal.

8. The whole is greater than its part.

9. Every whole is equal to all its parts taken together.

10. All right angles are equal to each other.

11. Angles that have equal measures, or arcs, are equal.

12. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.

13. Two straight lines cannot enclose a space.

## REMARKS.

1 A problem is something proposed to be done.

2. The perimeter of a figure is the sum of all its sides taken together.

3. The sum of any two sides of a triangle is greater than the third side.

4. In any triangle the sum of the three angles is equal to two right angles.

5. Every triangle is half the parallelogram which has the same base and the same altitude.

6. An angle inscribed in a semicircle is a right angle.

7. All angles in the same segment of a circle are equal to each other.

8. Triangles that have all the three angles of the one respectively equal to all the three angles of the other, are called equiangular triangles, or similar triangles.

9. In similar triangles the like sides, or sides opposite to the equal angles, are proportional.

10. The areas or spaces of similar triangles are to each other as the squares of their like sides.

11. The areas of circles are to each other as the squares of their diameters, radii, or circumference.

12. Similar figures are such as have the same number of sides, and the angles contained by their sides respectively equal.

13. The areas of similar figures are to each as the squares of their like sides.

14. If three quantities are proportional, the middle one is repeated, and the first is to the second as the second is to the third. In such a case the middle quantity is a mean proportional between the other two, and the last is a third proportional to the first and second; but, if there are four proportional quantities, the last is called a fourth proportional to the other three

## INSTRUMENTS.

§ **2.** The principal instruments used in describing or constructing geometrical figures are as follows :

### THE DIVIDERS OR COMPASSES.

The plain compasses consist of two inflexible rods of brass, revolving upon an axis at the vertex, and furnished with steel points.

## THE PARALLEL RULER.

The parallel ruler consists of two flat pieces of ebony, connected together with brass bars, having their extremities equidistant, by which contrivance, when the ruler is opened, the sides necessarily move in parallel lines.

## THE SCALE OF EQUAL PARTS.


The scale of equal parts consists of a certain number of equal portions of any convenient length, the extreme one or the left hand being subdivided into ten equal parts, and is called the unit of the scale, and the rest being numbered 1, 2, 3, &c.

In most scales an inch is taken for a common measure, and what an inch is divided into is generally set at the end of the scale.

This scale is used in laying down any distance, as inches, feet, chains, miles, &c. The several divisions may be considered as feet, for example, the decimal subdivision would be tenths of a foot. So also each of the principal divisions may be regarded as ten inches, ten feet, &c., and in this case the decimal subdivision will represent inches, feet, &c., respectively. This scale is limited to two figures, or any number less than 100 may be readily taken; but if the number should consist of three places of figures, the value of the third figure cannot be exactly ascertained, and in this case it is better to use a diagonal scale, by which any number consisting of three places of figures may be exactly found.

Let it be required to take from the scale a line equal to five inches and eight-tenths.

Place one foot of the dividers at 5 on the right, and extend the other to 8, which makes the eighth of the small divisions. The dividers will then embrace the required distance.

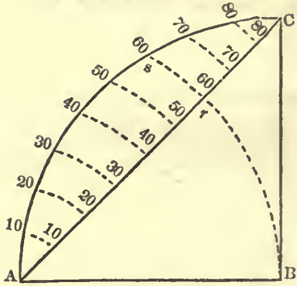
THE DIAGONAL SCALE OF EQUAL PARTS.


The construction of this scale is as follows:

Having prepared a ruler of convenient breadth for your scale, draw near the edges thereof two right lines, *a e*, *c f*, parallel to each other; divide one of these lines, as *a e*, into equal parts, according to the size of your scale, and through each of these divisions draw right lines perpendicular to *a e*, to meet *c f*; then divide the breadth into ten equal parts, and through each of these divisions draw right lines parallel to *a e* and *c f*; divide the lines *a b*, *c d*, into ten equal parts, and from the point *a*, to the first division in the line *c d*, draw a diagonal line; then parallel to that line, draw diagonal lines through all the other divisions, and the scale is complete. Then, if any number consisting of three places of figures, as 468, be required from the larger scale, *f d*, you must place one foot of the compasses on the figure 4, on the line *f d*, then the extent from 4 to the point *d* will represent 400. The second figure being 6, count six of the smaller divisions from *d* towards *c*, and the extent from 4 to that point will be 460. Move both points of the compasses downwards till they are on the eighth parallel line below *f d*, and open them a little till the one point rests on the vertical line drawn through 4, and the other on the diagonal line drawn through 6; the extent, then, in the compasses will represent 468. In the same manner the quantities 46.8, 4.68, 0.468, &c., are measured.

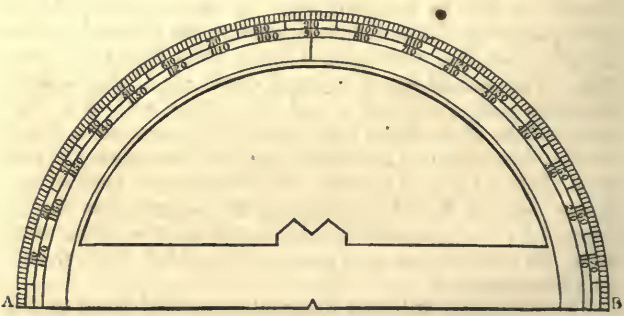
There are generally two diagonal scales laid down on the same face of the instrument, the unit of the one being double that of the other, and commencing on opposite ends of the scale.

### THE SCALE OF CHORDS.




Draw the lines B A, B C, at right angles to each other, and with any convenient distance, B A, describe the arc A s C; divide it into 90 equal parts, and join A C. From A as a centre, with the distances A 10, A 20, &c., describe the arcs 10, 10, 20, 20, &c., meeting the line A C. Fill up the separate degrees, which are not marked in the diagram to prevent confusion, and the scale is complete. It is evident, by inspection, that the chord of 60 is equal to the radius, as shown by the letter *r* upon the rule; which distance is therefore always to be taken in laying down angles.

### THE SEMICIRCULAR PROTRACTOR.




The protractor is a semicircular piece of brass divided into 180 degrees, and numbered each way from end to end; that is, from A to B, and from B to A. There is a small notch

in the middle of the diameter A B, denoting the centre of the protractor. In some boxes of mathematical instruments this is omitted, and the degrees are transferred to the border of the plain scale.

### GUNTER'S SCALE.

Gunter's scale, commonly of two feet in length, contains on one side the lines of the plain scale, already described, and on the other corresponding logarithmic lines.

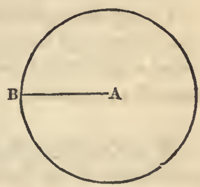
### PROBLEM I.

§ **3.** To describe from a given centre the circumference of a circle having a given radius.

Let A be the given centre, and A B the given radius.

Place one foot of the dividers at A, and extend the other leg until it shall reach to B.

Then turn the dividers around the leg at A, and the other leg will describe the required circumference.




### PROBLEM II.

Through a given point A, to draw a line parallel to a given line, D E.

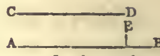
Lay the edge of the parallel ruler upon D E, and move it upwards till it reaches the point A, through which draw BC, and it will be parallel to DE.

### PROBLEM III.

To lay off on a given line, as A B, a distance equal to C D.

#### EXAMPLE.

Let C D be the distance to be laid off, and A B the given line.




Place one foot of the dividers at C, and extend the other leg until the foot reaches D.

Then, raising the dividers, place one foot at A, and mark with the other the distance A E, this will evidently be equal to C D.

### PROBLEM IV.

To lay down a line of given length, on a scale of a given number to the inch, to determine how many parts of it are to be represented on the paper by a distance equal to the unit of the scale.

#### EXAMPLE.

If a line 320 feet in length is to be laid down on paper, on a scale of 40 feet to the inch, what length must be taken from the scale?

Divide the length of the line by the number of parts which is represented by the unit of the scale; the quotient will give the number of parts which is to be taken from the scale.

Here $320 \div 40 = 8$ the number of parts to be taken from the scale.

### PROBLEM V.

The length of the line being given on the paper, to determine the true length of the line which it represents.

#### EXAMPLE.

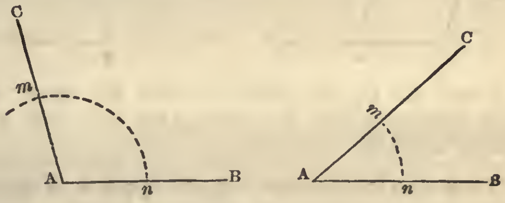
The length of the line on the paper is 4.75 inches, and the scale is one of 20 feet to the inch; what is the true length of the line?

Take the line in your dividers and apply it to the scale, and note the number of units and parts of a unit to which it is equal; then multiply this number by the number of parts which the unit of the scale represents, and the product will be the length of the line.

Here $4.75 \times 20 = 95$ feet, the length of the line.

### PROBLEM VI.

To make an angle of any proposed number of degrees.




1. Draw any line A B, and having taken first 60 degrees from the scale of chords, describe with this radius the arc $n\ m$.

2. Take in like manner the chord of the proposed number of degrees from the same scale, and apply it from $n$ to $m$.

3. Then if the line A C be drawn from the point A through $m$, the angle B A C will be that required.

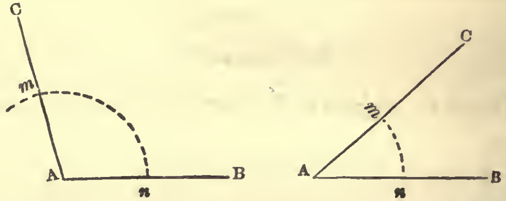
Angles greater than 90 degrees are usually made by first laying off 90 degrees upon the arc $n\ m$, and then the remaining part.

This problem may be performed by the protractor.

Place the central notch of the instrument upon A, and the edge along A B; make a point $m$ against the proposed number of degrees, and through it draw the line A C.

## PROBLEM VII.

Any angle B A C being given, to find the number of degrees it contains.




1. From the angular point A, with the chord of 60 degrees, describe the arc *n m*, cutting the lines A B, A C in the points *n* and *m*.

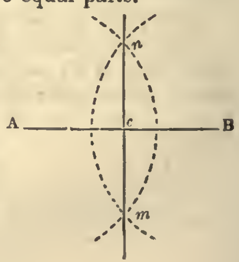
2. Then take the distance *n m*, and apply it to the scale of chords, and it will show the degrees required.

And if the distance *n m* be greater than 90°, it must be taken at twice, and each part applied separately to the scale. This problem may be performed by the protractor.

Place the central notch of the instrument upon A, and the edge along A C, and observe the number of degrees cut by the line A B, which will show the degrees required.

## PROBLEM VIII.

To divide a given line AB into two equal parts.

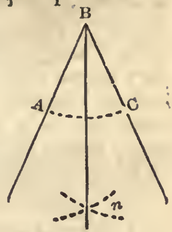



1. From the points A and B, as centres, with any distance greater than half A B, describe arcs cutting each other in *n* and *m*.

2. Through these points, draw the line *n c m*, and the point *c*, where it cuts A B, will be the middle of the line required.

## PROBLEM IX.

To divide a given angle A B C into two equal parts.



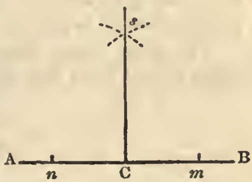

1. From the point B, with any radius, describe the arc A C; and from A and C, with the same, or any other radius, describe arcs cutting each other in *n*.

2. Then, through the point *n* draw the line B *n*, and it will bisect the angle A B C, as was required.

## PROBLEM X.

From a given point C, in a given right line A B, to erect a perpendicular.

Case I. When the point is near the middle of the line.



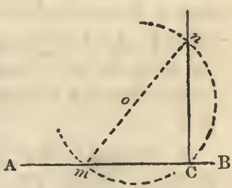

1. On each side of the point C take any two equal distances C*n*, C*m*.

2. From *n* and *m*, with any radius greater than *n* C or *m* C, describe arcs cutting each other in *s*.

3. Then through the point *s*, draw the line *s* C, and it will be the perpendicular required.

Case II. When the point is at, or near, the end of the line.

Supposing C to be the given point, as before.



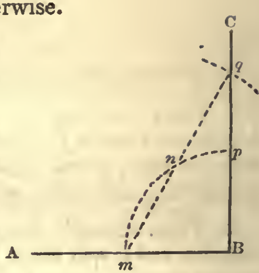

1. Take any point *o*, and with the radius or distance, *o* C, describe the arc *m* C *n*, cutting A B in *m* and C.

2. Through the centre *o*, and the point *m*, draw the line *m o n*, cutting the arc *m* C *n* in *n*.

3. Then from the point *n*, draw the line *n* C, and it will be the perpendicular required.

The same otherwise.

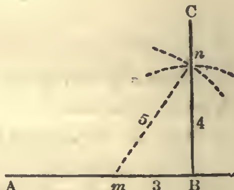
1. Set one leg of the compasses on B, and with any extent B*m* describe an arc *m p*; then set off the same extent from *m* to *n*.




2. Then join *m n*, and from *n* as a centre with the extent *m n* as radius, describe an arc *q*; produce *m n* to *q*, and the line joining *q* B will be perpendicular to A B.

Another method.

1. From any scale of equal parts take a distance equal to 3 divisions, and set it from B to *m*.




2. And from the points B and *m*, with the distances 4 and 5, taken from the same scale, describe arcs cutting each other in *n*.

3. Through the points *n*, B, draw the line B C, and it will be the perpendicular required.

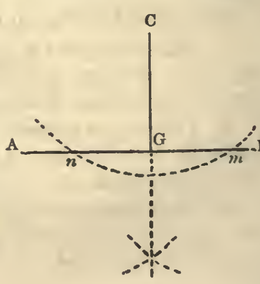
The same thing may also be readily done, by an instrument in the form of a square, or by the plain scale.

## PROBLEM XI.

From a given point C, without a given line A B, to let fall a perpendicular.

Case I. When the point is nearly opposite to the middle of the line.

1. From the point C, with any radius, describe the arc *n m*, cutting A B in *n* and *m*.

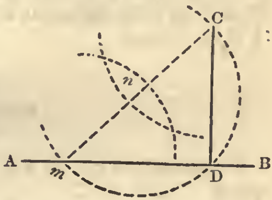



2. From the points *n*, *m*, with the same or any other radius, describe two arcs cutting each other in *s*.

3. Through the points C, *s*, draw the line C G *s*, and C G will be the perpendicular required

This problem, also, may be performed like the last by means of a square.

Case II. When the point is opposite or nearly opposite to the end of the line.




1. Take any point *m*, in the line A B, and from C draw the line C *m*.

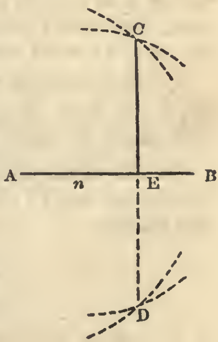
2. Bisect the line C *m*, or divide it into two equal parts in the point *n*.

3. From *n*, with the radius *nm*, or *n* C, describe the arc C D *m*, cutting A B in D.

4. Then through the point C, draw the line C D, and it will be the perpendicular required.

This method may also be used in the first case, if the line A B, when necessary, be produced.

The same otherwise.




1. From A, or any other point in A B, with the radius A C, describe the arcs C, D.

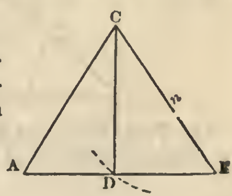
2. And from any other point *n*, in A B, with the radius *n* C, describe another arc cutting the former in C, D.

3. Then through the points C, D, draw the line C E D, and C E will be the perpendicular required.

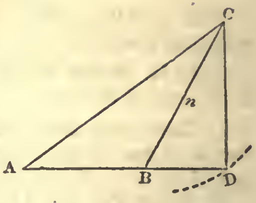
Perpendiculars may be more easily raised, and let fall, in practice, by means of a square, or other instrument proper for this purpose.

## PROBLEM XII.

To draw a perpendicular, from any angle of a triangle A B C, to its opposite side.



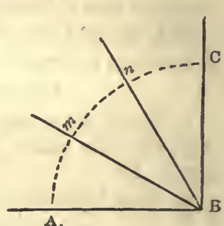

1. Bisect either of the sides containing the angle from which the perpendicular is to be drawn, as B C in the point *n*




2. Then with the radius $n$C, and from the centre $n$ describe an arc cutting AB, (or AB produced if necessary, as in the second figure,) in the point D; the line joining C D will be perpendicular to A B or A B produced.

## PROBLEM XIII.

To trisect, or divide a right angle A B C, into three equal parts.




1. From the point B, with any radius B A, describe the arc A C, cutting the legs B A, B C, in A, C.

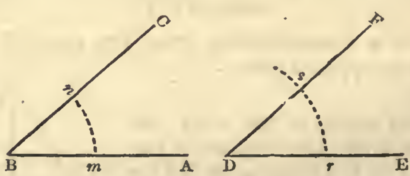
2. From the point A, with the radius A B, or B C, cross the arc A C in $n$; and with the same radius, from the point C, cross it in $m$.

3. Then through the points $m$, $n$, draw the lines B$m$, B$n$, and they will trisect the angle as was required.

By this means the circumference of any given circle may be divided into 12 equal parts; and thence by bisection into 24, 48, &c.

## PROBLEM XIV.

At a given point D, to make an angle equal to a given angle A B C.




1. From the point B, with any radius, describe the arc $n$ $m$, cutting the lines B A, B C, in the points $m$, $n$.

2. Draw the line D E, and from the point D, with the same radius as before, describe the arc *r s*.

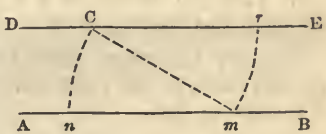
3. Take the distance *m n*, on the former arc, and apply it to the arc *r s*, from *r* to *s*.

4. Then through the points D, *s*, draw the line D F, and the angle E D F will be equal to the angle A B C, as was required.

## PROBLEM XV.

To draw a line parallel to a given line A B.

CASE I. When the parallel line is to pass through a given point C.

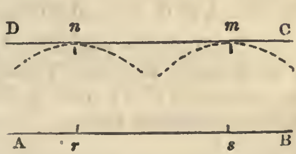



1. Take any point *m* in the line A B, and from the point C draw the line C *m*.

2. From the point *m*, with the radius *m* C, describe the arc C *n*, cutting A B in *n*; and with the same radius, from the point C, describe the arc *m r*.

3. Take the distance C *n*, and apply it to the arc *m r*, from *m* to *r*; then through the points C, *r*, draw the line D C *r* E, and it will be parallel to A B, as was required.

CASE II. When the parallel line is to be at a given distance from A B.

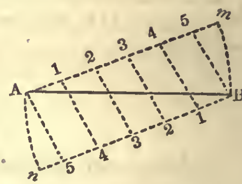



1. From any two points *r*, *s*, in the line A B, with a radius equal to the given distance, describe the arcs, *n*, *m*.

2. Then draw the line C D to touch these arcs without cutting them, and it will be parallel to A B, as was required

## PROBLEM XVI.

To divide a given line A B into any proposed number of equal parts.



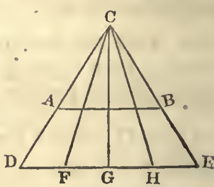

1. From one end of the line A, draw A $m$, making any angle with A B; and from the other end B, draw B $n$, making an equal angle A B $n$.

2. In each of the lines A $m$, B $n$, beginning at A and B, set off as many equal parts, of any convenient length, as A B is to be divided into.

3. Then join the points A 5, 1, 4, 2, 3, &c., and A B will be divided as was required.

B $n$ may be drawn parallel to A $m$, by means of the parallel ruler, or the same may be done by taking the arc A $n$ equal to B $m$.

### Another method.



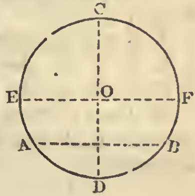

1. Through the point B draw the indefinite line C E, making an angle with A B.

2. Take any point E in that line, through which draw E D parallel to B A, and set off as many equal parts E H, H G, &c., from E toward D, as the line A B is to be divided into.

3. Through the points D, A, draw the line D A, producing it till it meets E C in C; then lines drawn from C through the points F, G, H, will divide the line A B, into the required number of parts.

### PROBLEM XVII.

To find the centre of a given circle, or of one already described.

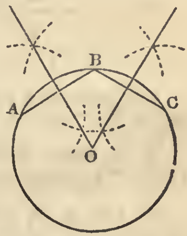



1. Draw any chord, A B, and bisect it with the perpendicular, C D.

2. Bisect C D, in like manner, with the chord E F, and the intersection, O, will be the centre required; observing that the bisection of the chords, and the raising of the perpendicular, may be performed as in Problems VIII. and X.

### PROBLEM XVIII.

To describe the circumference of a circle through any three given points, A, B, C, provided they are not in a right line.



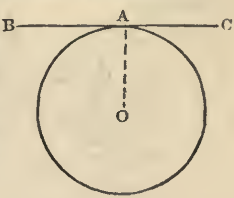

1. From the middle point B, draw the lines, or chords, B A, B C; and bisect them perpendicularly with the lines meeting each other in O.

2. Then from the point of intersection, O, with the distance O A, O B, or O C, describe the circle A B C, and it will be that required.

### PROBLEM XIX.

To draw a tangent to a given circle, that shall pass through a given point A.

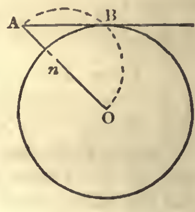
Case I.—When the point A is in the circumference of the circle.




1. From the given point A, to the centre of the circle, draw the radius A O.

2. Then through the point A, draw B C perpendicular to O A, and it will be the tangent required.

Case II.— When the given point, A, is without the circle.



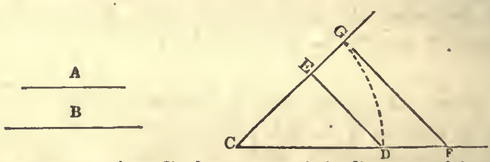

1. To the given point A, from the centre O, draw the line O A, and bisect it in $n$.

2. From the point $n$, with the radius $n$ A, or $n$ O, describe the semicircle A B O, cutting the given circle in B.

3. Then through the points A,B, draw the line A B, and it will be the tangent required.

### PROBLEM XX.

To two given right lines, A,B, to find a third proportional.




1. From any point, C, draw two right lines, making any angle, F C G.

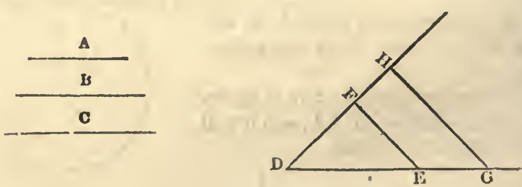
2. In these lines take C E equal to the first term A, and C G, C D, each equal to the second term B.

3. Join E D, and draw G F parallel to it; and C F will be the third proportional required.

That is C E (A) : C G (B) : : C D (B) : C F.

### PROBLEM XXI.

To three given right lines, A,B,C, to find a fourth proportional.



1. From any point, D, draw two right lines, making any angle G D H.

2. In these lines take D F equal to the first term A, D E equal to the second B, and D H equal to the third C.

3. Join F E, and draw H G parallel to it, then D G will be the fourth proportional required.

That is D F (A) : D E (B) : : D H (C) : D G.

## PROBLEM XXII.

Between two given right lines A, B, to find a mean proportional.


1. Draw any right line, in which take C E, equal to A, and E D, equal to B.

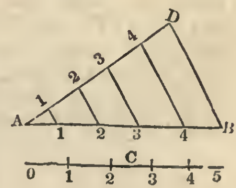
2. Bisect C D in O, and with O D or O C, as radius, describe the semicircle C F D.

3. From the point E draw E F perpendicular to C D, and it will be the mean proportional required.

That is C E (A) : E F : : E F : E D (B).

## PROBLEM XXIII.

To divide a given line, A B, in the same proportion that another given line, C, is divided.




1. From the point A draw A D equal to the given line C, and making any angle with A B.

2. To A D apply the several divisions of C, and join D, B.

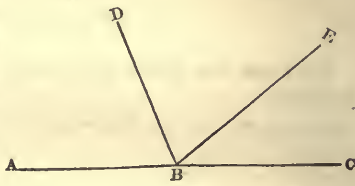
3. Draw the lines 4,4, 3,3, &c., each parallel to D B, and the line A B will be divided as was required.

That is, the parts A, 1; 1, 2; 2, 3; 3, 4; 4, B; on the line A B, will be proportional to the parts 0, 1; 1, 2; 2, 3; 3, 4; 4, 5; on the line C.

## PROBLEM XXIV.

Two angles of a triangle being given to find the third.

1. Draw the indefinite line A B C; at any point, as B, make the angle A B D equal to one of the given angles, and the angle D B E, equal to the other.

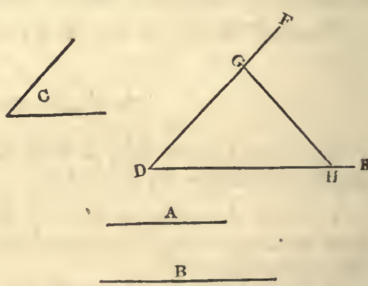



2. The remaining angle E B C will be the third angle required, because these three angles are together equal to two right angles.

## PROBLEM XXV.

Two sides of a triangle, and the angle which they contain, being given to construct the triangle.

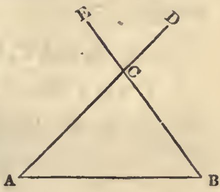
1. Let A and B be equal to the given sides, and C the given angle.




2. Draw the indefinite line D E; at the point D, make the angle E D F equal to the given angle C; then take D G=A, D H=B, and draw G H; then D G H will be the triangle required.

## PROBLEM XXVI.

A side and two angles being given to construct the triangle.

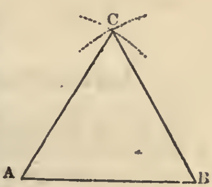



1. The two angles will either be adjacent to the given side, or the one adjacent and the other opposite ; in the latter case, find the third angle by subtracting the sum of the two given angles from 180 degrees ; then the two adjacent angles will be known.

2. Draw the line A B equal to the given side ; at the point A, make an angle, B A D, equal to one of the adjacent angles, and at B, an angle equal to the other ; the two lines A D, B E, will cut each other at C ; and A B C will be the triangle required.

## PROBLEM XXVII.

Upon a given right line, A B, to make an equilateral triangle.




1. From the points A and B, with a radius equal to A B, describe arcs cutting each other in C.

2. Draw the lines A C, B C, and the figure A C B will be the triangle required.

An isosceles triangle may be formed in the same manner, by taking any distance for radius.

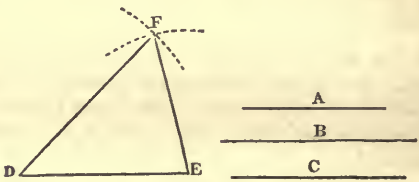
## PROBLEM XXVIII.

The three sides of a triangle being given, to describe the triangle.

Let A, B, and C, be the sides.

1. Draw a line, D E, equal to one of the given lines, C.

2. From the point D, with a radius equal to B, describe an arc.

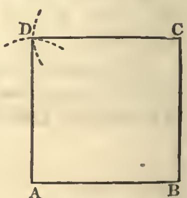



3. And from the point E, with a radius equal to A, describe another arc, cutting the former in F.

4. Then draw the lines D F, E F, and D F E will be the triangle required.

## PROBLEM XXIX.

Upon a given line, A B, to describe a square.




1. From the point B, draw B C perpendicular, and equal to A B.

2. From the points A and C, with the radius A B or C B, describe two arcs cutting each other in D; then draw the lines A D, C D, and the figure A B C D will be the square required.

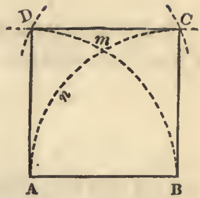
A rhombus may be made on the given line A B, in exactly the same manner, if B C be drawn with the proper obliquity, instead of perpendicularly.

Another method.

1. From the points A, B, as centres, with the radius A B, describe arcs crossing each other in *m*.

2. Bisect A *m* in *n*; and from the centre *m*, with the radius *m n*, cross the other two arcs in C and D.

3. Then draw the lines A D, B C, and C D, and A B C D will be the square required.

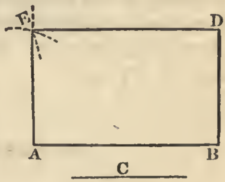



PROBLEM XXX.

To describe a rectangle, whose length and breadth shall be equal to two given lines, A B, and C.

1. At the point B, in the given line A B, erect the perpendicular B D, and make it equal to C.

2. From the points D, A, with the radii A B and C respectively, describe two arcs cutting each other in E; then join E A and E D, and A B D E will be the rectangle required.



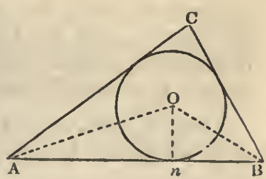

A parallelogram may be described in nearly the same manner.

PROBLEM XXXI.

In a given triangle, A B C, to inscribe a circle.

1. Bisect any two of the angles, as A and B, with the lines A O, and B O.

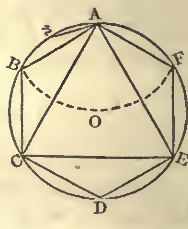
2. Then from the point of intersection, O, let fall the perpendicular O *n*, upon either of the sides, and it will be the radius of the circle required.



## PROBLEM XXXII.

In a given circle to inscribe an equilateral triangle, a hexagon, or a dodecagon.

### FOR THE HEXAGON.




1. From any point, A, in the circumference, as a centre, with a distance equal to the radius A O, describe the arc, B O F.

2. Join the points A, B, or A, F, and either of these lines being carried six times round the circle will form the hexagon required.

That is, the radius of the circle is equal to the side of the inscribed hexagon; and the sides of the hexagon divide the circumference of the circle into six equal parts, each containing 60 degrees.

### FOR THE EQUILATERAL TRIANGLE.

1. From the point A, to the second and fourth divisions, or angles of the hexagon, draw the lines A C, A E.

2. Then join the points C, E, and A C E will be the equilateral triangle required; the arcs, A B C, C D E, and E F A, being each one-third of the circumference, or 120 degrees.

### FOR THE DODECAGON.

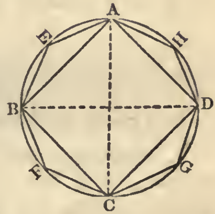
Bisect the arc A B, subtending the side of the hexagon in the point $n$, and the line A $n$ being carried twelve times round the circumference, will form the dodecagon required; the arc A $n$ being 30 degrees.

Then, if A $n$ be again bisected, a polygon may be formed of 24 sides; and by another bisection a polygon of 48 sides; and so on.

## PROBLEM XXXIII.

To inscribe a square or an octagon, in a given circle.

### FOR THE SQUARE.




1. Draw the diameters B D and A C, intersecting each other at right angles.

2. Then join the points A, B, B, C, C, A, and D, A; and A B C D will be the square required.

### FOR THE OCTAGON.

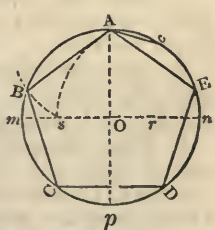
Bisect the arc A B, subtending the side of the square, in the point E, and the line A E being carried eight times round the circumference will form the octagon.

Also, if the arc A E be again bisected, a polygon may be formed of 16 sides: and by another bisection, a polygon of 32 sides; and so on.

## PROBLEM XXXIV.

To inscribe a pentagon, or decagon, in a given circle.

### FOR THE PENTAGON.




1. Draw the diameters A *p*, *n m*, at right angles to each other, and bisect the radius O *n* in *r*.

2. From the point *r*, with the distance *r* A, describe the arc A *s*, and from the point A, with the distance A *s*, describe the arc *s* B.

3. Join the points A, B, and the line A B being carried five times round the circle, will form the pentagon required.

### FOR THE DECAGON.

Bisect the arc A E, subtending the side of the pentagon in *c*, and the line A *c* being carried ten times round the circumference will form the decagon required.

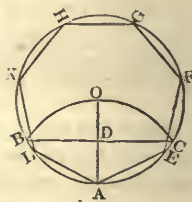
Also, if the arc A C be again bisected, a polygon of 20 sides may be formed; and by another bisection a polygon of 40 sides; and so on.

## PROBLEM XXXV.

In a given circle, to inscribe a regular heptagon.

1. From any point A in the circumference, with the radius A O of the circle, describe the arc B O C, cutting the circumference in B and C.

2. Draw the chord B C, cutting O A in D, and B D, or C D, carried seven times round the circle from A, will form the heptagon required.




An exact heptagon, as is well known, cannot be inscribed in a circle from pure geometrical principles; but, the above method of constructing it, which is extremely simple, will be found sufficiently accurate to answer most practical purposes; the approximation for the side being true, as far as the second place of decimals, inclusively.

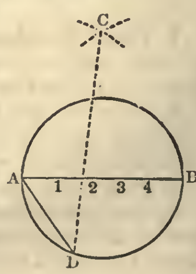
## PROBLEM XXXVI.

In a given circle, to inscribe any regular polygon.

1. Draw the diameter A B, which divide into as many equal parts as the figure has sides.

2. From the points A, B, as centres, with the radius A B, describe arcs crossing each other in C.

3. From the point C, through the second division of the diameter, draw the line C D; then if the points A and D be joined, the line A D will be the side of the polygon required.



It may be observed, that in the construction here given. A D is the side of a pentagon.

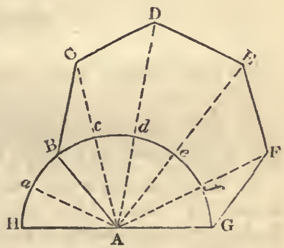
This rule is not exact, except for the equilateral triangle and hexagon; being for polygons in general only a near approximation.

## PROBLEM XXXVII.

On a given line A G, to describe a regular polygon of any proposed number of sides.

1. From the point A, with the distance A G, describe the semicircle G B H, which divide into as many equal parts as H *a*, *a* B, B *c*, &c., as the polygon is to have sides.

2. From A to the second part of the division draw A B, and through the other points *c*, *d*, *e*, *f*, &c., draw the lines A C, A D, A E, A F, &c.



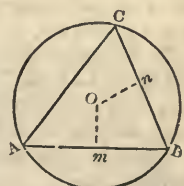

3. Apply the distance A G from G to F, from F to E, from E to D, from D to C, &c., and join B C, C D, D E. E F, &c., and A B C D E F G will be the polygon required.

## PROBLEM XXXVIII.

About a given triangle A B C, to circumscribe a circle.

1. Bisect the two sides A B, B C, with the perpendiculars *m* O, and *n* O.

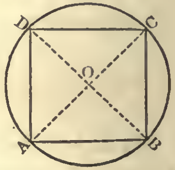
2. From the point of intersection O, with the distance O A, O B, or O C, describe the circle A C B, and it will be that required.




If any two of the angles be bisected, instead of the sides, the intersection of these lines will give, as before shown, the centre of the inscribed circle.

## PROBLEM XXXIX.

About a given square, A B C D, to circumscribe a circle.

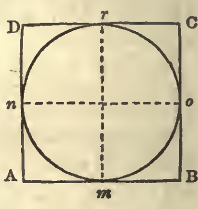



1. Draw the two diagonals A C and B D intersecting each other in O.

2. Then from the point O, with the distance O A, O B, O C or O D, describe the circle A B C D, and it will be that required.

## PROBLEM XL.

To circumscribe a square about a given circle.

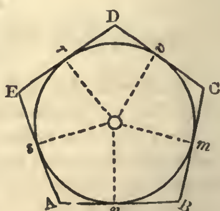



1. Draw any two diameters *n o* and *r m* at right angles to each other.

2. Then through the points *m*, *o*, *r*, *n*, draw the lines A B, B C, C D, and D A, perpendicular to *r m*, and *n o*, and A B C D will be the square required.

## PROBLEM XLI.

About a given circle to circumscribe a pentagon.




1. Inscribe a pentagon in the circle; or which is the same thing, find the points *m*, *n*, *v*, *r*, *s*, as in Prob. XXXIV.

2. From the centre *o*, to each of these points, draw the radii *o n*, *o m*, *o v*, *o r*, and *o s*.

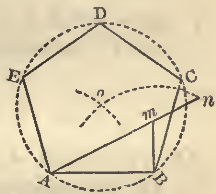
3. Through the points *n*, *m*, draw the lines A B, B C, perpendicular to *o n*, *o m*; producing them till they meet each other at B.

Draw in like manner the lines C D, D E, E A, perpendicular to *o v*, *o r*, and *o s*, and A B C D E will be the pentagon required.

Any other polygon may be made to circumscribe a circle, by first inscribing a similar one, and then drawing tangents to the circle at the angular points.

## PROBLEM XLII.

On a given line, A B, to make a regular pentagon.




1. Make B $m$ perpendicular to A B, and equal to one half of it.

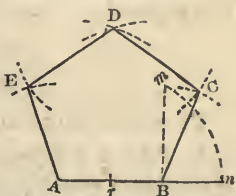
2. Draw A $m$, and produce it till the part $m$ $n$ is equal to B $m$.

3. From A and B as centres, with the radius B $n$, describe arcs cutting each other in $o$.

4. Then from the point $o$, with the same radius, or with $o$ A, or $o$ B, describe the circle, A B C D E; and the line A B, applied five times round the circumference of this circle, will form the pentagon required.

If tangents be drawn through the angular points A, B, C, D, E, a pentagon circumscribing the circle will be formed; and if the arcs be bisected, a circumscribing decagon may be formed; and so on.

### Another method.




1. Produce A B towards $n$, and at the point B make the perpendicular B $m$, equal to A B.

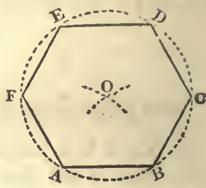
2. Bisect A B in $r$, and from $r$ as a centre, with the radius $r$ $m$, describe the arc $m$ $n$, cutting the produced line A B in $n$.

3. From the points A and B, with the radius A $n$, describe arcs cutting each other in D; and from the points A, D, and B, with the radius A B, describe arcs cutting each other in C and E.

4. Then if the lines B C, C D, D E, and E A, be drawn, A B C D E will be the pentagon required.

## PROBLEM XLIII.

On a given line, A B, to make a regular hexagon.

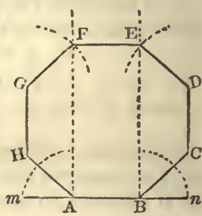



1. From the points A, and B, as centres, with the radius A B, describe arcs cutting each other in O; and from the point O, with the distance O A, or O B, describe the circle A B C D E F.

2. Then if the line A B be applied six times round the circumference, it will form the hexagon required.

## PROBLEM XLIV.

On a given line, A B, to form a regular octagon.




1. On the extremities of the given line, A B, erect the indefinite perpendiculars A F, and B E.

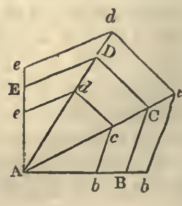
2. Produce A B both ways to *m* and *n*, and bisect the angles *m* A F and *n* B E with the lines A H and B C.

3. Make A H and B C each equal to A B, and draw H G, C D, parallel to A F, or B E, and also each equal to A B.

4. From G, and D, as centres, with a radius equal to A B, describe arcs crossing A F, B E, in F and E; then if G F, F E, and E D be drawn, A B C D E F G H will be the octagon required.

## PROBLEM XLV.

To make a figure similar to a given figure, A B C D E.



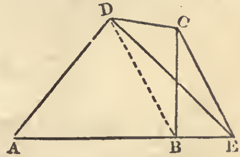

1. Take A *b*, equal to one of the sides of the figure required, and from the angle A draw the diagonals A *c*, A *d*.

2. From the points *b*, *c*, *d*, draw *b c*, *c d*, *d e*, parallel to B C, C D, D E, and A *b c d e* will be similar to A B C D E.

The same thing may also be done by making the angles *b*, *c*, *d*, *e*, respectively equal to the angles B, C, D, E.

### PROBLEM XLVI.

To make a triangle equal to a given trapezium, A B C D.

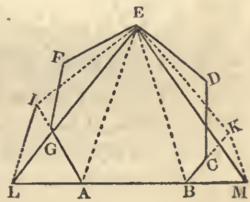



1. Draw the diagonal D B, and make C E parallel to it, meeting the side A B produced in E.

2. Join the points D, E, and A D E will be the triangle required.

### PROBLEM XLVII.

To make a triangle equal to any right-lined figure, A B C D E F G A.

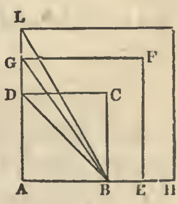



1. Produce the side A B both ways at pleasure.
2. Draw the diagonals E A, E B; and by the last problem, make the triangles A E I, B E K, equal to the figures A E F G, and B E D C.
3. Draw I L, K M, parallel to E A, E B.
4. Then if the points E, L, and E, M, be joined, E L M will be the triangle required.

And in the same manner may any right-lined figure whatever be reduced to a triangle.

### PROBLEM XLVIII.

To describe a square that shall be any multiple of a given square, A B C D.

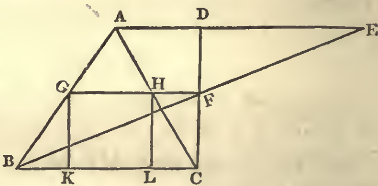



1. Draw the diagonal B D, and on A B, A D, produced, take A G, A E, each equal to B.D, then A F, the square on A E, will be double the square A C.
2. Draw in like manner the diagonal B G, and make A L, A H, each equal to B G; then A K, the square on A H, will be triple the square A C.

3. Proceed in the same manner with the diagonal B L; and a square will be formed which is quadruple the square A C; and so on.

### PROBLEM XLIX.

To inscribe a square in a given triangle, A B C.



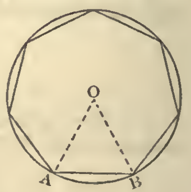

1. Through the vertex A, parallel to B C, draw the straight line A E, and from C raise a perpendicular to meet it in D.

2. Make D E equal to D C; join E B, cutting D C in F.

3. Through F draw F G, parallel to B C, and through H and G, draw H L, G K, parallel to D C. Then L H G K will be the square required.

### PROBLEM L.

In a given circle to inscribe a polygon of any proposed number of sides.




1. Divide 360° by the number of sides of the figure and make an angle A O B, at the centre, whose measure shall be equal to the degrees in the quotient.

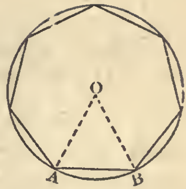
2. Then join the points A, B, and apply the chord A B to the circumference the given number of times, and it will form the polygon required.

### PROBLEM LI.

On a given line A B to form a regular polygon of any proposed number of sides.

1. Divide 360° by the number of sides of the figure, and subtract the quotient from 180 degrees.

2. Make the angles A B O and B A O each equal to half the difference last found; and from the point of intersection O, with the distance O A or O B, describe a circle.

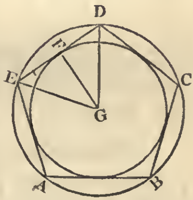



3. Then apply the chord A B to the circumference the proposed number of times, and it will form the polygon required.

## PROBLEM LII.

To describe a circle within or without a regular polygon.

1. Bisect any two angles, A E D, E D C, by the lines E G, D G, and from G let fall G F, perpendicular to the side E D.

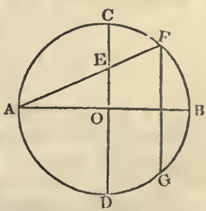



2. Then with the radius G E describe the outer circle, and with the radius G F describe the inner circle.

## PROBLEM LIII.

To make a square that shall be nearly equal to a given circle.

1. In the given circle A C B D, draw the two diameters A B, C D, cutting each other perpendicularly in the centre O.

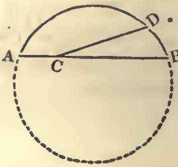



2. Bisect the radius O C, in E, and through the points A, E, draw the chord A E F, which will be the side of a square that is nearly equal in area to the circle.

If F G be drawn parallel to C D, it will be nearly equal to $\frac{1}{4}$ of the circumference of the circle.

## PROBLEM LIV.

To find a right line that shall be nearly equal to any given arc A D B of a circle.



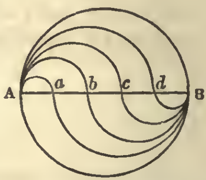

1. Divide the chord A B into four equal parts, and set one of the parts A C, on the arc from B to D.

2. Draw C D, and the double of this line will be nearly equal to the arc A D B.

If a right line be made equal to $3\frac{1}{7}$ times the diameter of a circle, it will be nearly equal to the circumference.

## PROBLEM LV.

To divide a given circle into any proposed number of parts, that shall be mutually equal to each other, both in area and perimeter.



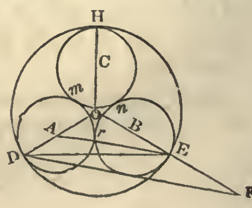

1. Divide the diameter A B into the proposed number of equal parts, at the points *a*, *b*, *c*, *d*.

2. On A*a*, A*b*, A*c*, A*d*, &c., as diameters, describe semicircles on one side of the diameter A B; and on B*d*, B*c*, B*b*, B*a*, &c., describe semicircles on the other side of the diameter.

3. Then the corresponding semicircles, joining each other as above, will divide the circle in the manner proposed.

## PROBLEM LVI.

In a given circle, to describe three equal circles which shall touch one another, and also the circumference of the given circle.



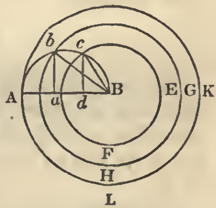

From the centre O, let the right-lines O H, O D, and O E be drawn, dividing the circumference into three equal parts in the points H, D, and E; join D E, and in O E produced take E F, equal to one

half of D E; draw D F, and parallel thereto draw E A meeting O D in A; make H C and E B each equal D A, and upon the centres A, B and C, through the points D, H, and E, let the circles A *r* D, C *m* H and B *n* E be described.

## PROBLEM LVII.

To divide a given circle into any number of equal parts by means of concentric circles.

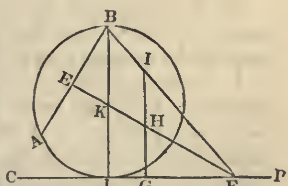
Let it be required to divide the circle A K L into three equal parts.




Divide the radius A B into three equal parts; and from the points of section *a*, *d*, draw the perpendiculars *a b*, *d c*, meeting the circumference of a semicircle described on A B, in *b* and *c*; and join B *c*, B *b*. Then, if circles be described from B, as a centre, with the radii B *c*, B *b*, the circle A K L will be divided into three equal parts, as required.

## PROBLEM LVIII.

To describe the circumference of a circle through two given points, A, B, which shall touch a right line, C D, whose position is given.



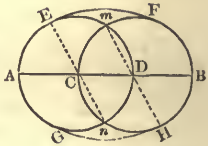

1. Draw the right line A B, joining the two given points, which bisect in E, by the perpendicular E F, meeting the given line C D in F.

2. Join B F, and from any point, H, in F E, let fall the perpendicular H G; and having made H I equal to H G, draw B K, parallel to I H, meeting F E in K.

3. Then, if a circle be described from the point K, as a centre, with the radius K B, it will pass through the points A and B, and touch the line C D, as was required.

## PROBLEM LIX.

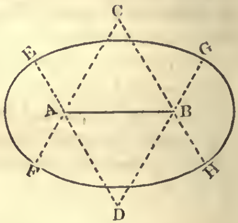
Upon a given line, A B, to describe an oval, or a figure resembling an ellipse.




1. Divide A B into three equal parts, A C, C D, D B; and from the points C, D, with the radii C A, D B, describe the circles A G D E, and C H B F.

2. Through the intersections $m$, $n$, and centres C, D, draw the lines $m$ H, $n$ E; and from the points $n$, $m$, with the radii $n$ E, $m$ H, describe the arcs E F, H G, and A G H B F E will be the oval required.

Another method.

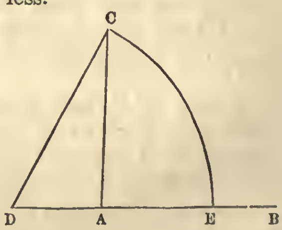



1. On A B, as a common base, describe the two equal isosceles triangles A C B, A D B, producing their sides C A, C B, D A, and D B.

2. From D and C, as centres, with any convenient radius, describe the arcs E G, F H; and from A and B, with the radius A E, B G, describe the arcs G H, E F, and the figure E F H G will be the oval required.

## PROBLEM LX.

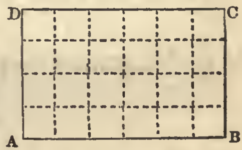
To divide a given straight line A B in extreme and mean ratio; that is so that the whole line shall be to the greater part as the greater part is to the less.




1. From A, draw A C at right angles to A B, and make A C = A B; produce B A to D, making A D = half A B; and about D, at the distance D C, describe the arc C E. Then A B is divided in extreme and mean ratio at the point E; and A B A E :: A E : E B.

# MENSURATION OF SUPERFICIES.

§ 4. The area of any figure is the measure of its surface, or of the space contained within the bounds of that surface, without any regard to the thickness. A square whose side is 1 inch, 1 foot, or 1 yard, &c., is called the measuring unit; and the area, or content of any figure, is estimated by the number of squares of this kind that are contained in it, as in the rectangle, ABCD.




## THE SQUARE.

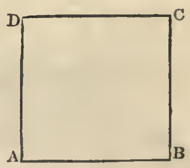
### PROBLEM I.

§ 5. To find the area of a square.

Rule.—Multiply the side by itself, and the product will be the area.

EXAMPLES.

1. What is the area of the square A B C D, whose side is 21 inches?




Here, $21 \times 21 = 441$ inches: the area required.

2. What is the area of a square whose side is 4 ft. 2 in.? *Ans.* 17 ft. 4 in. 4″.

3. What is the area of a square field whose side is 50 perches? *Ans.* 15 a. 2 r. 20 p.

4. What is the area of a square meadow whose side is 35.25 chains? *Ans.* 124 a. 1 r. 1 p.

5. How many yards are contained in a square whose side is 36 feet? *Ans.* 144 yds.

6. How many men can stand on 6 acres of land, each occupying a space of 3 feet square? *Ans.* 29040 men.

## PROBLEM II.

The area of a square being given, to find the length of the side.

RULE.—Extract the square root of the area.

### EXAMPLES.

1. The area of a square is 2025 feet; what is the side?

Here $\sqrt{2025} = 45$ ft. the side required.

2. What is the side of a square floor containing 729 square feet? *Ans.* 27 ft.

3. What is the side of a square whose area is 8 acres, 0 roods, 16 perches? *Ans.* 36 p.

4. What is the side of a square field whose area is 7 acres? *Ans.* 8.3666 chains.

5. Required the side of a square floor containing 1734 square feet? *Ans.* 41.6413 ft.

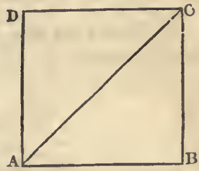
6. Required the side of a square meadow whose area is 6 acres, 2 roods, 14 perches? *Ans.* 32.4653 p.

## PROBLEM III.

The diagonal of a square being given, to find the area.

RULE.—Divide the square of the diagonal by 2, and the quotient will be the area.

EXAMPLES.




1. The diagonal of the square, A B C D, is 8 chains; what is the area?

Here $(8\times8)\div2=64\div2=32$ square chains, then $32\div10=3$ a. 0 r. 32 p.

2. The diagonal of a square is 12 yards; what is the area? *Ans.* 72 yds.

3. The diagonal of a square is 34 perches; what is the area? *Ans.* 3 a. 2 r. 18 p.

4. The diagonal of a square is 16 chains; what is the area? *Ans.* 12 a. 3 r. 8 p.

5. The diagonal of a square is 324 feet; what is the area? *Ans.* 52488 ft.

6. How many acres are contained in a square field whose diagonal is 29 chains? *Ans.* 42 a. 0 r. 8 p.

PROBLEM IV.

The area of a square being given, to find the diagonal.

RULE.—Extract the square root of double the area.

EXAMPLES.

1. The area of a square piece of land is 64 acres, 3 roods, 8 perches; what is the diagonal?

Here 64 a. 3 r. 8 p.$=10368$ perches; then $\sqrt{(10368\times2)}=\sqrt{20736}=144$ perches, the diagonal.

2. The area of a square is 578 feet; what is the diagonal? *Ans.* 34 ft.

3. The area of a square is 128 yards; what is the diagonal? *Ans.* 16 yds.

4. The area of a square field is 28.8 acres; what is the diagonal in chains? *Ans.* 24 chs.

5. The area of a square meadow is 16.2 acres; what is the diagonal in chains? *Ans.* 18 chains.

6. The area of a square is 4 acres and 8 perches; what is the diagonal? *Ans.* 36 perches.

## PROBLEM V.

The diagonal of a square being given, to find the side.

RULE.—Extract the square root of half the square of the diagonal.

### EXAMPLES.

1. The diagonal of a square is 24 yards; what is the side? Here $\sqrt{[(24\times 24)\div 2]}=\sqrt{(576\div 2)}=\sqrt{288}=16.9705$ yds.

2. The diagonal of a square is 18 feet; what is the side? *Ans.* 12.7279 feet.

3. The diagonal of a square is 36 chains; what is the side? *Ans.* 25.4558 chains.

4. What is the side of a square whose diagonal is 48 perches? *Ans.* 33.9411 perches.

5. What is the side of a square whose diagonal is 12 feet? *Ans.* 8.4852 feet.

6. What is the side of a square piece of land whose diagonal is 58 perches? *Ans.* 41.0121 perches.

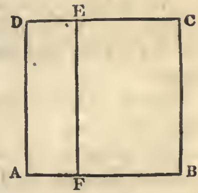
## PROBLEM VI.

To cut off a given area from a square, parallel to either side.

RULE.—Divide the given area by the length of the side, the quotient will be the length of the other side to be cut off.

### EXAMPLES.

1. What length must be cut from the square ABCD, whose sides are 25 chains, to have an area BCEF, of 40 acres, at the end?




Here 400 ch. $\div$ 25 = 16 chains, length required.

2. The sides of a square are 17 feet; what must be the length of another side to give an area of 153 square feet?

*Ans.* 9 feet.

3. The side of a square being 40 rods, what must be the length of the other side, to give $4\frac{1}{2}$ acres?

*Ans.* 18 rods.

4. The sides of a square are 15 chains; what must be the length of the other side to have an area of $7\frac{1}{2}$ acres?

*Ans.* 5 chains.

5. What length must be cut off from a square field whose sides are 125 perches to have an area of 50 acres?

*Ans.* 64 perches.

6. The area of a square piece of land is 29 acres, 3 r. 1 p.; what length must be cut off from the same to give an area of 10 acres and .8 perches? *Ans.* 23.2 perches.

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## THE RECTANGLE.

### PROBLEM I.

§ **6.** The length and breadth of a rectangle being given to find the area.

Rule.—Multiply the length by the breadth, and the product will be the area.

#### EXAMPLES.

1. What is the area of the rectangle A B C D, whose length A B is 14.5 feet, and breadth B C 11.6 feet?


Here, $14.5 \times 11.6 = 168.2$ feet, area required.

2. What is the area of a rectangle whose length is 14 feet 6 inches, and breadth 4 feet 9 inches? *Ans.* 68 ft. $10\frac{1}{2}$ in.

3. How many acres are contained in a rectangular piece of land whose sides are 46 and 58 chains? *Ans.* 266 acres, 3 r. 8 p.

4. How many square feet are contained in 28 boards, each 18 feet long and 16 inches wide? *Ans.* 672 feet.

5. How many squares of 100 feet each are contained in a floor 48 feet long and 21 feet wide? *Ans.* 10.08 squares.

6. What is the area of a rectangular piece of land whose length is 204.7 and breadth 117.8 yards? *Ans.* 24113.66 yards.

### PROBLEM II.

The area and either side of a rectangle being given, to find the other side.

Rule.—Divide the area by the given side, and the quotient will be the other side.

#### EXAMPLES.

1. The area of a rectangle is 456 feet, and the length 30 feet; what is the breadth?

Here $456 \div 30 = 15.2$ feet, the breadth.

2. The area of a rectangle is 846 chains, and its length 42 chains; what is the breadth? *Ans.* $20\frac{1}{7}$ chains.

3. The area of a rectangular piece of land is 6 a. 2 r. 16 p., and breadth $29\frac{1}{3}$ p.; what is the length?

*Ans.* 36 perches.

4. The area of a rectangle is 392 yards, and the shortest side 12 yards; what is the longest side?

*Ans.* $32\frac{2}{3}$ yards.

5. The area of a rectangular meadow is 73 acres, 3 roods 20 perches, and the length 120 perches; what is the breadth?

*Ans.* $98\frac{1}{2}$ p.

6. The area of a rectangle is 1728 feet, and the breadth 36 feet; what is the length? *Ans.* 48 ft.

## PROBLEM III.

The area and the proportion of the two sides of a rectangle being given, to find the sides.

RULE.—Multiply the area by the greater number of the proportion, and divide the product by the less; the square root of the quotient will be the length: then multiply the length by the less number of the proportion, and divide the product by the greater; the quotient will give the breadth.

### EXAMPLES.

1. The area of a rectangular piece of land is 432 acres, and the length is to the breadth as 5 to 3; what are the sides?

Here 432 acres $=$ 69120 perches.

Then $\sqrt{[(69120\times5)\div3]}=\sqrt{(345600\div3)}=\sqrt{115200}$ $=339.41125$ perches, the length;

And $(339.41125\times3)\div5=203.64675$ perches, the breadth.

2. The area of a rectangle is 1472 yards, and the breadth is to the length as 3 to 4; required the sides.

*Ans.* 33.2264, and 44.3019 yds.

3. The area of a rectangle is 24 acres, and the length is to the breadth as 3 to 2; what are the sides?

*Ans.* 18.9736, and 12.6491 chains.

4. The area of a rectangle is 27 acres, 3 roods, 20 perches, and the length is to the breadth as 9 to 7; required the sides.

*Ans.* 75.725, and 58.897 p.

5. The area of a rectangle is 28 acres, and the breadth is to the length as 4 to 7; required the sides.

*Ans.* 12.6491, and 22.1359 chains.

6. It is required to lay out 1 acre in a rectangular form, having the length to the breadth in the ratio of 8 to 5.

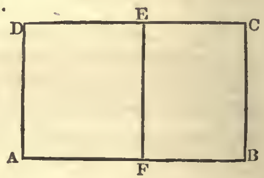
*Ans.* 88, and 55 yds.

## PROBLEM IV.

The sides of a rectangle being given, to cut off a given area parallel to either side.

RULE.—Divide the area by the side which is to retain its length or breadth, and the quotient will be the length or breadth of the other side.

### EXAMPLES.




1. The sides of the rectangle, A B C D, are 18.16, and 12.15 chains; what must be the length to leave an area, BCEF, of 12 acres adjoining the breadth?

Here 120 chs. $\div$ 12.15 = 9.8765 chains, the length.

2. The sides of a rectangle are 24 and 16 chains; what must be the breadth to leave 18 acres adjoining the length?

*Ans.* $7\frac{1}{2}$ chs.

3. The sides of a rectangle are 216 and 124 feet; what must be the length to leave 10106 square feet adjoining the breadth?

*Ans.* $81\frac{1}{2}$ ft.

4. The sides of a rectangle are 180 and 75 perches; what must be the breadth so as to leave $22\frac{1}{2}$ acres adjoining the length?

*Ans.* 20 p.

5. The sides of a rectangle are 8 and 15 yards; what must be the length to leave 46 square yards at the shortest side?

*Ans.* $5\frac{3}{4}$ yds.

6. The length of a rectangle is 14.5 chains, and the breadth 6.4 chains; what must be the breadth, the length being the same, to contain 5.8 acres?

*Ans.* 4 chs.

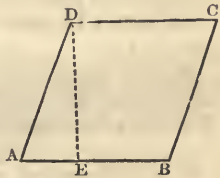
## THE RHOMBUS.

### PROBLEM I.

§ 7. To find the area of a rhombus.

RULE.—Multiply the length by the perpendicular height, and the product will be the area.

EXAMPLES.




1. The length of a rhombus, A B, is $18\frac{1}{2}$ feet, and the perpendicular height, D E, $7\frac{3}{4}$ feet; required the area.

Here, $18.5 \times 7.75 = 143.375$ feet.

2. Required the area of a rhombus whose length is 14 feet 4 inches, and its height 12 feet 2 inches.

*Ans.* 174 ft. 4 in. 8″.

3. What is the area of a rhombus whose length is 12 feet 3 inches, and height 9 feet 4 inches? *Ans.* 114 ft. 4 in.

4. Required the area of a rhombus whose length is 38 perches, and height 17 perches. *Ans.* 4 a. 0 r. 6 p.

5. What is the area of a rhombus whose length is 19 chains, and height 15 chains? *Ans.* 28 a. 2 r.

6. Required the area of a rhombus whose length is 27 yards, and height 21.5 yards. *Ans.* 580.5 yds.

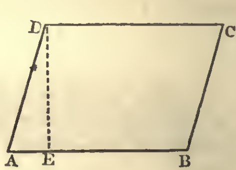
## THE RHOMBOID.

### PROBLEM I.

§ 8. To find the area of a rhomboid.

Rule.—Multiply the length by the perpendicular height, and the product will be the area.

#### EXAMPLES.




1. The length of a rhomboid, A B, is 28 perches and height, D E, 12; required the area.

Here $28 \times 12 = 336$ perches $= 2$ a. 0 r. 16 p.

2. The length of a rhomboid is 22 feet 9 inches, and height 14 feet 3 inches; how many square yards does it contain?

*Ans.* 36.0208 yds.

3. The length of a rhomboid is 16.2 yards, and height 9.6 yards; how many square perches does it contain?

*Ans.* 5.1411 sq. p.

4. The length of a rhomboid is 21 chains, and the height 11.5 chains; what is the area? *Ans.* 24 a. 0 r. 24 p.

5. How many acres are contained in a rhomboid whose length is 130 perches and height 57 perches?

*Ans.* 46 a. 1 r. 10 p.

6. How many square yards are contained in a rhomboid whose length is 271 feet and height 107 feet?

*Ans.* $3221\frac{8}{9}$ sq. yds.

### PROBLEM II.

The area of a rhombus or rhomboid, and the length of the side being given, to find the perpendicular height; or the area and the height being given, to find the length of the side.

Rule.—Divide the area by the length of the side, and the quotient will be the perpendicular height; or divide by the height, and the quotient will be the length of the side.

EXAMPLES.

1. The area of a rhombus is 27 perches, and the length of the side 6.75 perches; what is the perpendicular height?

Here $27 \div 6.75 = 4$ perches.

2. The area of a rhombus is 4 acres, and the height 5 chains; what is the length of the side?

*Ans.* 8 chains.

3. Required the length of a rhombus whose area is 17 acres, and height 35 perches. *Ans.* $77\frac{5}{7}$ perches.

4. The area of a rhomboid is 4 acres, 3 roods, 18 perches, and the length of the side 38 perches; what is the height?

*Ans.* $20\frac{9}{19}$ perches.

5. The area of a rhomboid is 1776 square feet, and the height 24 feet; what is the length? *Ans.* 74 feet.

6. The area of a rhomboid is 36 acres, and the length of the sides 24 chains; what is the height?

*Ans.* 15 chains.

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## THE TRIANGLE.

### PROBLEM I.

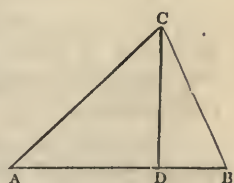
§ **9.** To find the area of a triangle, when the base and perpendicular height are given.

RULE.—Multiply the base by the perpendicular height, and half the product will be the area.

EXAMPLES.

1. What is the area of a triangle ABC, whose base, AB, is 46 feet, and height, DC, 23 feet?

Here $(46 \times 23) \div 2 = 1058 \div 2 =$ 529 feet, the area.




2. What is the area of a triangle whose base is 67 yards, and height is 14.5 yards? *Ans.* 485.75 yds.

3. What is the area of a triangle whose base is 207.5 feet, and height is 59.5 feet? *Ans.* 6173.125 feet.

4. The base of a triangle is 72 perches, and the height $12\frac{1}{2}$ perches; how many acres does it contain?

*Ans.* 2 a. 3 r. 10 p.

5. What is the area of a triangle whose base is 12.25 chains, and the height 8.5 chains? *Ans.* 5 a. 0 r. 33 p.

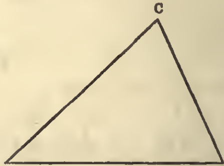
6. What is the area of a triangular field whose base is $24\frac{1}{2}$ chains, and the height 18 chains? *Ans.* 22 a. 0 r. 8 p.

### PROBLEM II.

The three sides of a triangle being given, to find the area.

RULE.—From half the sum of the three sides, subtract each side severally. Then multiply the half sum, and the three remainders continually together, and the square root of the product will be the area required.

#### EXAMPLES.




1. What is the area of a triangle A B C, whose three sides, B C, C A, A B, are 23.7, 29.25 and 40.1 yards?

Here $(23.7 + 29.25 + 40.1) \div 2 = 93.05 \div 2 = 46.525 =$ half the sum of the sides.

$46.525 - 23.7 = 22.825 =$ first difference.
$46.525 - 29.25 = 17.275 =$ second difference.
$46.525 - 40.1 = 6.425 =$ third difference.

Whence $\sqrt{(46.525 \times 22.825 \times 17.275 \times 6.425)} = \sqrt{117865.94866835} = 343.3161$ yards, the area.

2. What is the area of a triangular field whose sides are 26, 28 and 30 chains? *Ans.* 33 a. 2 r. 16 p.

3 Required the area of an equilateral triangle whose side is 22 perches. *Ans.* 1 a. 1 r. 9.5781 p.

4. Required the area of an isosceles triangle whose base is 30, and each of its equal sides 45 feet?

*Ans.* 636.3961 feet.

5. What is the area of a triangle whose sides are 22.2, 38, and 40.1 feet? *Ans.* 413.7114 feet.

6. What is the area of a triangular field whose sides are 27.35, 31.15, and 38 chains? *Ans.* 42 a. 0 r. 6.6955 p.

## PROBLEM III.

Any two sides of a right-angled triangle being given, to find the third side.

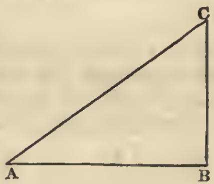
Rule 1st.—To the square of the perpendicular add the square of the base, and the square root of the sum will give the hypothenuse.

2d.—The square root of the difference of the squares of the hypothenuse, and either side will give the other.

3d. Or multiply the sum of the hypothenuse and either side by their difference, and the square root of the product will give the other.

### EXAMPLES.

1. The base of a right-angled triangle, A B, is 27 yards, and the perpendicular, B C, 36 yards; what is the hypothenuse?




Here $\sqrt{(36^2 + 27^2)} = \sqrt{(1296 + 729)} = \sqrt{2025} = 45$ yards, the hypothenuse.

2. The hypothenuse of a right-angled triangle is 242 feet, and the perpendicular 182 feet; what is the base?

Here $\sqrt{(242^2 - 182^2)} = \sqrt{(58564 - 33124)} = \sqrt{25440} = 159.4989$ feet, the base.
Or, $\sqrt{[(242 + 182) \times (242 - 182)]} = \sqrt{(424 \times 60)} = \sqrt{25440} = 159.4989$ feet, the base as before.

3. The base of a right-angled triangle is 38 chains, and the perpendicular 41 chains; required the hypothenuse. *Ans.* 55.9016 chs

4. The hypothenuse of a right-angled triangle is 68 perches, and the perpendicular 33 perches; what is the base? *Ans.* 59.4558 p.

5. The hypothenuse of a right-angled triangle is 315 feet, and the base 299 feet; required the perpendicular.
*Ans.* 99.1160 feet.

6. The top of a May pole, being broken off by a blast of wind, struck the ground at 9 feet distance from the foot of the pole; what was the height of the whole May pole, supposing the length of the broken piece to be 41 feet?
*Ans.* 81 ft.

7. Two ships sail from the same port, one east 60 miles, and the other north 80 miles; how far are they apart?
*Ans.* 100 m.

8. A line 78 yards long will reach from the top of a fort, on the opposite bank of a river, to the water edge on this side of the river; what is the height of the fort, the river being 76 yards across? *Ans.* 17.5499 yds.

9. A ladder of 100 feet in length was placed against a building of 100 feet high, in such a manner that the top of it reached within six inches of the top of the building; what was the distance of the foot of the ladder from the base of the edifice? *Ans.* 9.9874 ft.

10. A ladder 30 feet long, placed near the middle of a street, reached the buildings at one side 24 feet from the ground; and the opposite side, without moving the foot, 18 feet: what was the breadth of the street? *Ans.* 42 ft.

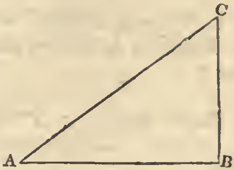
## PROBLEM IV.

The sum of the hypothenuse and perpendicular, and the base of a right-angled triangle being given, to find the hypothenuse and the perpendicular.

Rule.—To the square of the sum add the square of the base, and divide the number by twice the sum of the hypothenuse and perpendicular, and the quotient will be the hypothenuse.

Subtract the hypothenuse from the sum of the hypothenuse and perpendicular, and the remainder will be the perpendicular.

EXAMPLES.




1. The sum of the hypothenuse and perpendicular is 100 feet, and the base 40 feet; required the hypothenuse and the perpendicular.

Here $(100^2 + 40^2) \div (100 \times 2) = (10000 + 1600) \div 200 = 11600 \div 200 = 58$ feet, the hypothenuse. And $100 - 58 = 42$ feet, the perpendicular.

2. The height of a tree, standing perpendicularly on a plane, is 110 feet. At what height must it break off, so that the top may rest on the ground 50 feet from the base, and the place broken on the upright part? *Ans.* $43\frac{7}{11}$ ft.

3. The base of a right-angled triangle is 36 chains, and the sum of the hypothenuse and perpendicular is 84 chains; what is the hypothenuse? *Ans.* $49\frac{5}{7}$ ch.

4. A tree 90 feet high, growing perpendicularly on a plane, was broken off by the wind; the broken part resting on the upright, and the top on the ground 30 feet from the base; what was the length of the broken part?

*Ans.* 50 ft.

5. The sum of the hypothenuse and perpendicular is 240 yards, and the base 80 yards; required the perpendicular.

*Ans.* $106\frac{2}{3}$ yds.

6. A May pole, whose height was 84 feet, standing on a horizontal plane, was broken by the wind, and the extremity of the top part struck the ground at the distance of 24 feet from the bottom of the pole; required the length of each part.

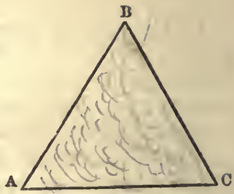
*Ans.* $45\frac{3}{7}$ feet, the hypoth. and $38\frac{4}{7}$ the perp.

## PROBLEM V.

To determine the area of an equilateral triangle.

Rule.—Multiply the square of the side by .433013, and the product will be the area.

EXAMPLES.




1. What is the area of an equilateral triangle, A B C, whose side is 20 feet?

Here $(20)^2 \times .433013 = 400 \times .433013 = 173.2052$ feet, the area.

2. What is the area of an equilateral triangle whose side is 40 feet? *Ans.* 692.8208 feet.

3. What is the area of an equilateral triangle whose side is 80 perches? *Ans.* 17.3205 acres.

4. What is the area of an equilateral triangle whose side is 24.4 yards? *Ans.* 257.7986 yards.

5. How many acres are contained in an equilateral triangle whose side is 16 chains? *Ans.* 11 a. 0 r. 13.6212 p.

6. How many acres are contained in an equilateral triangle whose side is 32 perches? *Ans.* 2 a. 3 r. 3.4053 p.

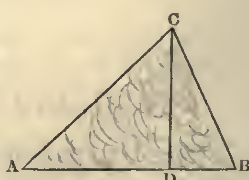
## PROBLEM VI.

The area and the base of any triangle being given, to find the perpendicular height.

Or, the area and height being given, to find the base.

RULE.—Divide double the area by the base, and the quotient will be the perpendicular height; or divide double the area by the height, and the quotient will be the base.

EXAMPLES.




1. The area of a triangle, A B C, is 4 a. 3 r. 32 p., and the base, A B, 24 perches; what is the perpendicular height, D C?

Here 4 a. 3 r. 32 p. = 792 p. and $(792 \times 2) \div 24 = 1584 \div 24 = 66$ perches, the height.

2. The area of a triangle is 806.3125 yards, and the perpendicular 33.25 yards; what is the base?

$(806.3125 \times 2) \div 33.25 = 1612.625 \div 33.25 = 48.5$ yards, the base.

3. The area of a triangle is $102\frac{1}{2}$ feet, and the base 20 feet; what is the perpendicular? *Ans.* $10\frac{1}{4}$ feet.

4. The area of a triangle is 5 a. 0 r. 33 p. and the perpendicular 28.5 p.; what is the base? *Ans.* 58.4561 perches.

5. If the area of a triangle be $862\frac{16}{35}$ yards, and its base $17\frac{1}{5}$ yards; what is its perpendicular height?

*Ans.* $100\frac{2}{7}$ yards.

6. If the area of a triangular field be 4.39775 acres, and its perpendicular 7.18 chains; what is its base?

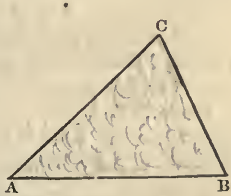
*Ans.* 12.25 chains.

## PROBLEM VII.

The proportion of the three sides of a triangle being given, to find the sides of a triangle corresponding with a given area.

RULE.—Find the area of the triangle according to the given proportion, by Problem II. page 68; then as that area is to the area given, so is the square of either of its sides to the square of the similar side; the square root of which will be the required side. The other sides of the triangle will be proportional to the corresponding given sides.

### EXAMPLES.




1. A person wishes to enclose 6 acres, 1 rood, 12 perches, in a triangle, similar to a small triangle whose sides are 6, 8, and 9 perches respectively; required the sides of the triangle.

Here 23.5252 perches = the area of the triangle, whose sides are 6, 8, and 9.

6 a. 1 r. 12 p. = 1012 perches.

As 23.5252 : 1012 :: 64 : 2753.1328, and $\sqrt{}$ 2753.1328 = 52.47 perches, one of the sides.

Now as 8 : 9 :: 52.47 : 59.029 p. } the other two sides.
" " 8 : 6 :: 52.47 : 39.353 p. }

2. The area of a triangle is 24 acres; what must be the length of the sides, in the proportion 3, 4 and 5 chains?

*Ans.* 18.973, 25.298, and 31.622 chains.

3. The area of a triangle is 10 acres; what must be the length of the sides, in the proportion of 5, 11 and 13 chains?

*Ans.* 9.64, 21.208 and 25.064 chains.

4. What are the sides of a triangle containing one acre, in the proportion of 3, 4 and 6 chains?

*Ans.* 4.1082, 5.4776 and 8.2164 chains.

5. What are the sides of a triangle containing 33.6 acres, in the proportion of 13, 14 and 15 chains?

*Ans.* 26, 28 and 30 chains.

6. What are the sides of a triangle containing 24 acres, in the proportion of 5, 12 and 13 chains?

*Ans.* 14.1421, 33.9411 and 36.7695 chains.

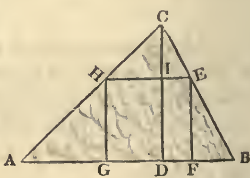
## PROBLEM VIII.

The base and perpendicular of any plane triangle being given, to find the side of its inscribed square.

Rule.—Divide the product of the base and perpendicular by their sum, and the quotient will be the side of the inscribed square

### EXAMPLES.

1. The base A B of a triangle is 12 feet, and the perpendicular, C D, 18 feet; what is the side, EF, of the inscribed square?




Here $(12 \times 18) \div (12 + 18) = 216 \div 30 = 7.2$ feet, $=$ EF, the side of the inscribed square

2. The base of an isosceles triangle is 24 yards, and the perpendicular 16 yards; what is the side of the inscribed square?

*Ans.* 9.6 yards.

3. The hypothenuse of a right-angled triangle is 16 perches, and the base 14 perches; what is the side of the inscribed square?

*Ans.* 4.9868 p.

4. The base of a scalene triangle is 20 chains, and the perpendicular 15 chains; what is the side of the inscribed square? *Ans.* $8\frac{4}{7}$ chs.

5. The area of a scalene triangle is 32 acres, and the base 25 chains; required the side of the inscribed square.
*Ans.* 12.6482 chs.

6. The area of an isosceles triangle is 324 feet, and the perpendicular 18 feet; required the side of the inscribed square. *Ans.* 12 ft.

## PROBLEM IX.

The three sides of any triangle being given, to find the length of a perpendicular which will divide it into two right-angled triangles.

RULE.—Upon the base, let fall a perpendicular from the opposite angle; this perpendicular will divide the base into two parts called segments, and the whole triangle into two right-angled triangles.

Then, as the base or sum of the segments is to the sum of the other two sides, so is the difference of these sides to the difference of the segments of the base.

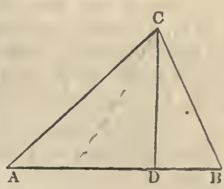
To half the base add half the difference of the segments, and the sum will be the greater segment; also from half the base subtract half the difference of the segments, and the remainder will be the less segment.

When the perpendicular falls without the triangle, the base is to the sum of the sides as the difference of the sides is to the sum of the segments of the base.

Then in each of the two right-angled triangles, there will be known the hypothenuse and base; consequently, the perpendicular may be found by Problem III., page 69.

### EXAMPLES.

1. In the triangle, A B C, are given, A B 20, A C 18, and B C 14 yards; required the perpendicular, C D.




Here, As 20 : (18 + 14) : : (18 — 14) : 6.4, the difference of the segments of the base: then, 6.4 ÷ 2 = 3.2 yards, half their difference, and

$20 \div 2 = 10$ yards, half the base; now $10 + 3.2 = 13.2$ yards, the segment A D, and $10 - 3.2 = 6.8$ yards, the segment B D. Therefore, $\sqrt{(BC^2 - BD^2)} = \sqrt{(14^2 - 6.8^2)} = \sqrt{(196. - 46.24)} = \sqrt{149.76} = 12.2376$ yards, the perpendicular, D C.

2. The base of a triangle is 426 feet, and the other two sides 365 and 230 feet; required the length of the perpendicular. *Ans.* 196.9904 ft.

3. The base of a triangle is 80 chains, and the other two sides 60 and 40 chains; what is the length of the perpendicular which divides it into two right-angled triangles? *Ans.* 29.0473 chs.

4. The base of a triangle is 128 perches, and the other two sides 94 and 68 perches; required the length of the perpendicular. *Ans.* 48.6137 p.

5. The base of a triangle is 324 yards, and the other two sides 264 and 162 yards; required the length of the perpendicular. *Ans.* 131.2613 yds.

6. The base of a triangle is 30 perches, and the other two sides 24 and 18 perches; required the length of the perpendicular. *Ans.* 14.4 p.

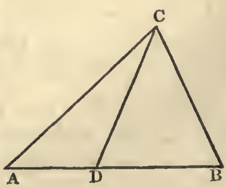
## PROBLEM X.

The area and base of a triangle being given, to cut off a given part of the area by a line running from the angle opposite the base.

RULE.—As the given area of the triangle is to the area of the part to be cut off, so is the given base to the base corresponding to that area.

### EXAMPLES.

1. Given the area of a triangle, A B C, 12 acres, and the length of the base, A B, 24 chains; it is required to cut off 5 acres towards the angle A, by a line running from the angle C to the base.




Here 12 acres = 120 chains, and 5 acres = 50 chains.

Then, as A B C (120) : A D C (50) : : A B (24) : A D (10 chains.)

2. In the triangle A B C, there are given the area 54 acres, 2 roods, 30 perches, and the base, A B, 70 perches, to cut off 20 acres towards the angle B, by a line, C D, running from the angle C to the base; required the part B D of the base. *Ans.* 25.6 p.

3. In the triangle A B C, there are given the area $7\frac{1}{2}$ acres, and the base, A B, 8 chains, to cut off $1\frac{7}{8}$ acres towards the angle A, by a line, C D, running from the angle C to the base; required the part A D of the base. *Ans.* 2 chs.

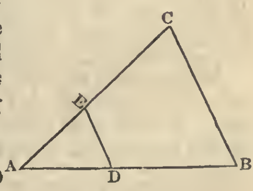
## PROBLEM XI.

The area and base of a triangle being given, to cut off a triangle containing a given area, by a line running parallel to one of its sides.

Rule.—As the given area of the triangle is to the area of the triangle to be cut off, so is the square of the given base to the square of the required base. The square root of the result will be the base of the required triangle.

### EXAMPLES.

1. Given the area of the triangle A B C, 250 square chains, and the base, A B, 20 chains; it is required to cut off 60 square chains towards the angle A, by a line, D E, running parallel to B C.




As A B C (250) : A D E (60) : : A $B^2$ (400) : A $D^2$ (96). And A D $= \sqrt{}$ A $D^2$ (96) $= 9.7979$ chains.

2. Given the area of a triangle, A B C, 20 acres, and the base, A B, 50 chains, to find D B, a part of the base, so that a line, D E, running from the point D, parallel to the side. A C, may cut off a triangle, B D E, containing 9 acres. *Ans.* B D = 33.541 chs.

3. Given the area of a triangle, A B C, 5 acres, and the

base, A B, $12\frac{1}{2}$ perches, to find A D, a part of the base, so that a line, D E, running from the point D, parallel to the side B C, may cut off a triangle, A D E, containing $2\frac{1}{4}$ acres.

*Ans.* A D = 8.3852 p.

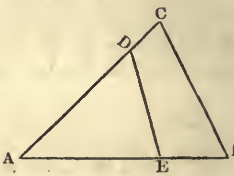
### PROBLEM XII.

The area and two sides of a triangle being given, to cut off a triangle containing a given area, by a line running from a given point in one of the given sides, and falling on the other.

RULE.—As the given area of a triangle is to the area of the part to be cut off, so is the rectangle of the given sides to a fourth term.

Divide this fourth term by the distance of the given point from the angular point of the two given sides; the quotient will be the distance of the required point from the same angle.

#### EXAMPLES.




1. Given the area of a triangle, A B C, $2\frac{1}{2}$ acres; the side A B, 25 perches; the side A C, 20 perches; and the distance of a point D, from the angle A, 18 perches; it is required to find a point, E, to which, if a line be drawn from the point D, it shall cut off a triangle, A D E, containing 1 acre, 2 roods, 10 perches.

Here, as the area of the triangle A B C, 400 sq. p. : the area of the triangle A D E, 250 sq. p. : : A B $\times$ A C (25 $\times$ 20) : A D $\times$ A E (312.5). Then A D $\times$ A E (312.5) $\div$ A D (18) = A E = 312.5 $\div$ 18 = 17.36 per.

2. Given the area of a triangle, A B C, 24.7875 acres; the side A B, 40 chains; the side A C, 32.5 chains; and the distance of a point E, in the side A B, from the angle A, 17 chains; it is required to find the distance A D, in the line A C, so that a line drawn from E to D may cut off a triangle A E D, containing 6 acres. *Ans.* 18.51 chs.

## THE TRAPEZIUM.

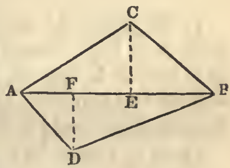
### PROBLEM I.

§ **10.** To find the area of a trapezium.

Rule.—Multiply the sum of the two perpendiculars by the diagonal upon which they fall, from the opposite angles and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezium, A C B D, whose diagonal, A B, is 84 yards, the perpendicular C E, 28 yards, and the perpendicular D F, 21 yards.




Here $[(28 + 21) \times 84] \div 2 =$ $(49 \times 84) \div 2 = 4116 \div 2 = 2058$ yards, the area of the trapezium A C B D.

2. Required the area of a trapezium whose diagonal is 33 perches, and the perpendiculars 11 perches and 13 perches. *Ans.* 2 a. 1 r. 36 p.

3. How many acres are there in the trapezium whose diagonal is 80.5 chains, and the perpendiculars 22.4 and 28.3 chains? *Ans.* 204 a. 0 r. 10.8 p.

4. What is the area of a trapezium whose diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches and 60 feet 9 inches? *Ans.* 6347.25 ft.

5. How many square yards of paving are there in a trapezium whose diagonal is 65 feet, and the two perpendiculars let fall on it, from its opposite angles, 28 and $33\frac{1}{2}$ feet respectively? *Ans.* 222.083 yds.

6. How many acres are there in the trapezium whose diagonal is 4.75 chains, and the two perpendiculars falling on it, from its opposite angles, 2.25 and 3.6 chains respectively? *Ans.* 1 a. 1 r. 22.3 p.

## THE TRAPEZOID.

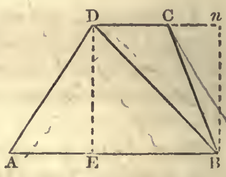
### PROBLEM I.

§ 11. To find the area of a trapezoid.

Rule.—Multiply the sum of the two parallel sides by the perpendicular distance between them, and half the product will be the area.

#### EXAMPLES.

1. Required the area of the trapezoid, A B C D, whose parallel sides, D C and A B, are 14 feet 6 inches and 24 feet 9 inches, and the perpendicular distance, D E, 8 feet 3 inches.




Here $[(A\,B + D\,C) \times D\,E] \div 2 = [(24.75 + 14.5) \times 8.25] \div 2 = (39.25 \times 8.25) \div 2 = 323.8125 \div 2 = 161.90625$ feet, the area.

2. How many square feet are there in a plank 1 foot 6 inches broad at one end, and 1 foot 3 inches at the other, the length being 20 feet? *Ans.* $27\frac{1}{2}$ ft.

3. Required the area of a trapezoid whose parallel sides are 24.46 chains, and 38.4 chains, and the perpendicular distance 16.2 chains. *Ans.* 50 a. 3 r. 26.6 p.

4. Required the area of a trapezoid whose two parallel sides are 25 feet 6 inches, and 18 feet 9 inches, and the perpendicular distance between them 10 feet 5 inches. *Ans.* $230\frac{15}{32}$ ft.

5. The two parallel sides of a trapezoid are 12.41 and 8.22 chains, respectively, and the perpendicular distance between them 5.15 chains; required the area. *Ans.* 5 a. 1 r. 9.956 p.

6. Required the area of a trapezoid whose two parallel sides are 750 and 1225 links, and the perpendicular distance between them 1540 links. *Ans.* 15 a. 0 r. 33.2 p.

## POLYGONS.

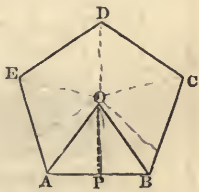
### PROBLEM I.

§ **12.** To find the area of a regular polygon.

RULE.—Multiply the perimeter, or sum of all the sides of the figure, by the perpendicular falling from its centre upon one of the sides, and half the product will be the area.

EXAMPLES.

1. Required the area of the regular pentagon, ABCDE, one of whose equal sides, AB, or BC, &c., is 25 yards, and the perpendicular OP from its centre, 17.2 yards.




Here $(25 \times 5 \times 17.2) \div 2 = 2150 \div 2 = 1075$ yards, the area.

2. The side of a regular hexagon is 15 feet, and the perpendicular 13 feet; what is the area? *Ans.* 585 feet.

3. Required the area of a regular hexagon whose side is 14.6 yards, and perpendicular 12.64 yards?
*Ans.* 553.632 yds.

4. How many acres are contained in a regular heptagon whose sides are each 19.38 chains, and perpendicular from the centre 20 chains? *Ans.* 135.66 acres.

5. The side of a regular pentagon is 10 yards, and the perpendicular is 6.882 yards; what is the area?
*Ans.* 172.05 yards.

6. What is the area of a regular octagon whose sides are each 19.882 feet, and the perpendicular from the centre 24 feet? *Ans.* 1908.672 feet.

### PROBLEM II.

To find the area of a regular polygon when one of its equal sides only is given.

RULE.—Multiply the square of the side of the polygon, by the number standing opposite to its name in the following table, and the product will be the area.

*Table, when the side of the polygon is* 1.

| No. of sides. | Names. | Areas, or Multipliers. | Radius of inscribed circle |
|---|---|---|---|
| 3 | Trigon or equil. △ | 0.433013 | 0.288675 |
| 4 | Tetragon or square | 1.000000 | 0.500000 |
| 5 | Pentagon | 1.720477 | 0.688191 |
| 6 | Hexagon | 2.598076 | 0.866025 |
| 7 | Heptagon | 3.633912 | 1.038262 |
| 8 | Octagon | 4.828427 | 1.207107 |
| 9 | Nonagon | 6.181824 | 1.373739 |
| 10 | Decagon | 7.694209 | 1.538842 |
| 11 | Hendecagon | 9.365640 | 1.702844 |
| 12 | Duodecagon | 11.196152 | 1.866025 |

EXAMPLES.

1. The side of a regular pentagon is 12 feet; what is the area?

Here $12^2 \times 1.720477 = 144 \times 1.720477 = 247.748688$ feet, the area.

2. What is the area of a regular octagon, the side being 20 feet? *Ans.* 1931.3708 ft.

3. The side of a regular hexagon is 24 feet; what is its area? *Ans.* 1496.4917 ft.

4. What is the area of a regular heptagon whose side is 16 yards? *Ans.* 930.2814 yds.

5. What is the area of a regular nonagon whose side is 36 inches? *Ans.* 8011.6439 in.

6. How many pieces, each 4 inches square, may be cut from a regular decagon whose side is 12 inches? *Ans.* 69.2478 pieces.

### PROBLEM III.

When the area of any regular polygon is given, to find the side.

Rule.—Divide the area by the number in the table corresponding with the figure, and the square root of the quotient will be the length of the side.

#### EXAMPLES.

1. The area of a regular pentagon is 4 a res; how many perches are contained in the side?

Here $160 \times 4 = 640$ perches,
Then $\sqrt{(640 \div 1.720477)} = \sqrt{371.9898} = 19.2870$ perches, the length of the side.

2. Required the length of the side of a regular hexagon containing one acre. *Ans.* 7.8475 perches.

3. The area of an octagonal floor is 560 feet; what is the length of the side? *Ans.* 10.7693 feet.

4. The area of a regular hexagon is 73.9 feet; what is the side? *Ans.* $5\frac{1}{3}$ feet.

5. The area of a regular heptagon is 1356.6 yards; what is the length of the side? *Ans.* 19.3214 yards.

6. The area of a regular decagon is 3233.4912 feet; what is the length of the side? *Ans.* $20\frac{1}{2}$ feet.

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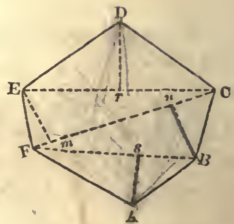
## IRREGULAR FIGURES.

### PROBLEM I.

§ **13.** To find the area of an irregular right-lined figure of any number of sides.

Rule 1.—Divide the figure into triangles and trapeziums, and find the areas of each of them separately by Prob. I. p. 67, and Prob. I. p. 79; then add these areas together, and their sum will give the area of the whole figure.

### EXAMPLES.




1. Required the area of the irregular right-lined figure, A B C D E F, the dimensions of which are as follows: F B = 20.75, F C = 27.48. E C = 18.5, B $n$ = 14.25, E $m$ = 9.35, D $r$ = 12.8, and A $s$ = 8.6 perches respectively.

Here (F B $\times$ A $s$) $\div$ 2 = (20.75 $\times$ 8.6) $\div$ 2 = 178.45 $\div$ 2 = 89.225 perches, the area of the triangle, A B F.

And (E C $\times$ D $r$) $\div$ 2 = (18.5 $\times$ 12.8) $\div$ 2 = 236.8 $\div$ 2 = 118.4 perches, the area of the triangle, D E C.

Also, [(B$n$ + E$m$) $\times$ F C] $\div$ 2 = [(14.25 + 9.35) $\times$ 27.48] $\div$ 2 = (23.6 $\times$ 27.48) $\div$ 2 = 648.528 $\div$ 2 = 324.264 perches, the area of the trapezium, F B C E.

Whence 324.264 + 89.225 + 118.4 = 531.889 perches = 3 a. 1 r. 11.889 p., the area of the whole figure.

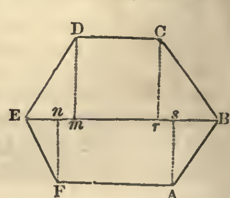
2. Required the area of an irregular hexagon, like that in the last example, supposing the dimensions of the different lines to be the halves of those before given.

*Ans.* 3 r. 12.9722 p.

RULE 2.—The area of any irregular right-lined figure may also be determined, by drawing perpendiculars from all its angles, to one of its diagonals, considered as a base; and then adding the areas of all the triangles and trapezoids together for the content.

### EXAMPLES.

1. Required the area of the irregular right-lined figure, A B C D E F, the dimensions of which are as follows: E $n$ = 4.54, E $m$ = 8.26, E $r$ = 20.01, E $s$ = 26.22, E B = 30.15, D $m$ = 10.56, C $r$ = 12.24, F $n$ = 8.56, and A $s$ = 9.26 chains respectively.




Here $m\ r$ = E $r$ — E $m$ = 20.01 — 8.26 = 11.75, $n\ s$ = E $s$ — E $n$ = 26.22 — 4.54 = 21.68, B $r$ = E B — E $r$ = 30.15 — 20.01 = 10.14, and B $s$ = E B — E $s$ = 30.15 — 26.22 = 3.93.

Whence $(F\,n \times E\,n) \div 2 = (8.56 \times 4.54) \div 2 = 38.8624 \div 2 = 19.4312$ chains, area of the triangle E F $n$.

And $(D\,m \times E\,m) \div 2 = (10.56 \times 8.26) \div 2 = 87.2256 \div 2 = 43.6128$ chains, area of the triangle, E D $m$.

Also $(C\,r \times B\,r) \div 2 = (12.24 \times 10.14) \div 2 = 124.1136 \div 2 = 62.0568$ chains, area of the triangle C B $r$.

And $(A\,s \times B\,s) \div 2 = (9.26 \times 3.93) \div 2 = 36.3918 \div 2 = 18.1959$ chains, area of the triangle A B $s$.

Then $[(D\,m + C\,r) \times m\,r] \div 2 = [(10.56 + 12.24) \times 11.75] \div 2 = (22.8 \times 11.75) \div 2 = 267.9 \div 2 = 133.95$ chains, area of the trapezoid D C $r\,m$.

And $[(F\,n + A\,s) \times n\,s] \div 2 = [(8.56 + 9.26) \times 21.68] \div 2 = (17.82 \times 21.68) \div 2 = 386.3376 \div 2 = 193.1688$ chains, area of the trapezoid F A $s\,n$.

Hence $19.4312 + 43.6128 + 62.0568 + 18.1959 + 133.95 + 193.1688 = 470.4155$ chains $= 47$ a. 0 r. 6.648 p., the area of the whole figure.

2. Required the area of an irregular figure, like that in the last example, only doubling the dimensions of the diagonal, and the several perpendiculars. *Ans.* 188 a. 0 r. 26.592 p.

## PROBLEM II.

To find the area of a mixtilineal figure, or one formed by right lines and curves.

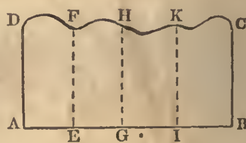
Rule 1.—Take the perpendicular breadths of the figure in several places, at equal distances from each other, and divide their sum by their number, for the mean breadth; and this quotient, multiplied by the length, will give nearly the true area of the figure.

2d. Or, if greater accuracy be required, take half the sum of the two extreme breadths, for one of the said breadths, and add it to the others, as before; then divide this sum by the number of parts in the base (instead of by the number of breadths), and the result multiplied by the length will give the area, with a sufficient degree of correctness to answer most questions of this kind that can occur. When the curved, or mixtilineal, boundary meets the base, as is

frequently the case in surveying, the area is found by dividing the sum of all the breadths by the number of parts in the base, and then multiplying the result by the length, as before; observing, in each of these cases, that the greater the number of parts into which the base is divided, the nearer will the approximation be to the exact area.

It may likewise be further remarked, that if the perpendiculars, or breadths, be not at equal distances from each other, the parts should be computed separately, as so many trapezoids, and then added together, for the area.

### EXAMPLES.




1. The perpendicular breadths of the irregular mixtilineal figure, A B C D, at 5 equidistant places, A E G I B, being 9.2, 10.5, 8.3, 9.4, and 10.7 yards, and its length, A B, 20 yards; what is its area?

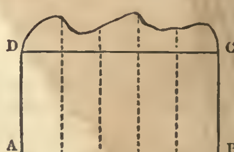
Here $(9.2 + 10.5 + 8.3 + 9.4 + 10.7) \div 5 = 48.1 \div 5 = 9.62$,

And $9.62 \times 20 = 192.4$ yards, area by the first part of the rule.

Or,

$(AD + BC) \div 2 = (9.2 + 10.7) \div 2 = 19.9 \div 2 = 9.95$.

Then $(9.95 + 10.5 + 8.3 + 9.4) \div 4 = 38.15 \div 4 = 9.5375$, and $9.5375 \times 20 = 190.75$ yards, the area by the second part of the rule.




2. Required the area of the figure, A B C D, of which the part A C is a rectangle, whose sides are $20\frac{1}{2}$ and $10\frac{1}{2}$ chains respectively; and the perpendicular breadths of the curvilineal spaces, reckoning from D C at 4 equidistant places, are 10.2, 8.7, 10.9, and 8. 5 chains respectively.

Here $(10.2 + 8.7 + 10.9 + 8.5) \div 5 = 38.3 \div 5 = 7.66$,

And $7.66 \times 20.5 = 157.03$ chains, the area of the curved space.

Then $20.5 \times 10.5 = 215.25$ chains, the area of the rectangle.

Whence $215.25 + 157.03 = 372.28$ chains $=$ 37 a 0 r 36.48 p., the area of the whole figure.

3. The length of an irregular mixtilineal figure is 47 chains, and its breadth, at 6 equidistant places, beginning at the left hand extremity of the base, 5.7, 4.8, 7.5, 5.1, 8.4, and 6.5 chains respectively; what is its area?

*Ans.* 29 a. 3 r. $2\frac{2}{3}$ p. by the first rule.

4. The length of an irregular mixtilineal figure, of which the curvilinear boundary meets the base, is $37\frac{1}{2}$ chains, and its breadth, at 7 equidistant places, is 4.9, 5.6, 4.5, 8.2, 7.3, 5.9, and 8.5 chains respectively; what is its area?

*Ans.* 24 a. 0 r. 8.571 p. by the first rule.
And 23 a. 3 r. 20 p. by the second rule.

5. The length of an irregular field is 39 rods, and its breadths, at five equidistant places, are 2.4, 2.6, 2.05, 3.65 3.6 rods respectively; what is the area?

*Ans.* 111.54 rods by the first rule.

6. The length of an irregular piece of land being 42 chains, and the breadths, at six equidistant points, being 8.7, 10.3, 7.1, 8.24, 10.04, 12.2 chains respectively; what is the area? *Ans.* 38 a. 2 r. 39.872 p., by the second rule.

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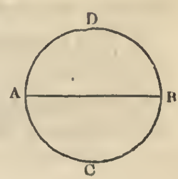
## THE CIRCLE.

### PROBLEM I.

§ **14.** To find the circumference of a circle, when the diameter is given, or the diameter, when the circumference is given.

Rule.—Multiply the diameter by 3.1416, and the product will be the circumference; or divide the circumference by 3.1416, or multiply the circumference by .31831, and the result will be the diameter.

#### EXAMPLES.




1. What is the circumference of the circle, A C B D, whose diameter, A B, is 7 feet?
Here $3.1416 \times 7 = 21.9912$ feet, the circumference.

2. What is the diameter of a circle whose circumference is 100 yards?

Here $100 \div 3.1416 = 31.831$ yards, the diameter.

Or,

$100 \times .31831 = 31.831$ yards, the diameter as before.

3. If the diameter of a circle be 17 chains; what is the circumference? *Ans.* 53.4072 chains.

4. If the circumference of a circle be 354 perches; what is the diameter? *Ans.* 112.6817 perches.

5. What is the circumference of the earth, supposing its diameter to be 7935 miles, which it is very nearly?

*Ans.* 24928.596 miles.

6. If the circumference of a carriage-wheel be 16 feet 6 inches; what is its diameter? *Ans.* 5.2521 feet.

## PROBLEM II.

To find the area of a circle.

RULE.—Multiply the square of the diameter by .7854; or the square of the circumference by .07958, and the product in either case will be the area.

### EXAMPLES.

1. How many square feet are there in a circle whose diameter is 5 feet 6 inches?

Here $(5.5)^2 \times .7854 = 30.25 \times .7854 = 23.75835$ sq. feet.

2. Required the area of a circle, the circumference of which is $9\frac{1}{5}$ yards.

Here $(9.2)^2 \times .07958 = 84.64 \times .07958 = 6.7356512$ sq. yds.

3. How many square yards are there in a circle whose radius is $15\frac{1}{4}$ feet? *Ans.* 81.1798 sq. yards.

4. How many square feet are there in a circle whose circumference is $10\frac{3}{4}$ yards? *Ans.* 82.7681 sq. feet.

5. The diameter of a circle is 16 chains; how many acres does it contain? *Ans.* 20 a. 0 r. 16.9984 p.

6. What is the value of a circular garden whose diameter is 6 perches, at the rate of 75 cents per square yard?

*Ans.* \$641.47$\frac{1}{2}$.

### PROBLEM III.

The area of a circle being given, to find the diameter or circumference.

RULE.—Divide the area by .7854, and the square root of the quotient will be the diameter. Or, divide the area by .07958, and the square root of the quotient will be the circumference.

#### EXAMPLES.

1. The area of a circle is 5 acres, 3 roods, and 26 perches; what is the diameter?

Here 5 a. 3 r. 26 p. = 946 perches; and $\sqrt{(946 \div .7854)} = \sqrt{1204.48179271} = 34.7056$ perches, the diameter.

2. The area of a circle being 2 acres, 3 roods, and 12 perches, what is the circumference?

Here 2 a. 3 r. 12 p. = 452 perches; and $\sqrt{(452 \div .07958)} = \sqrt{5679.69} = 75.3637$ perches, the circumference.

3. The area of a circle is $5028\frac{4}{7}$ square yards; what is its diameter? *Ans.* 80.0160 yds.

4. The area of a circular garden being 1 acre, what is the length of a stone wall which will enclose it?

*Ans.* 44.8392 p.

5. It is required to find the radius of a circle whose area is an acre. *Ans.* 39.2507 yds.

6. The value of a circular piece of ground, at $4 per square perch, is $46.50. How many dollars will encircle it if the diameter of a dollar be $1\frac{1}{2}$ inches?

*Ans.* $1595.3916.

### PROBLEM IV.

To find the area of a circular ring, or the space included between two concentric circles.

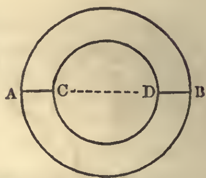
RULE.—Find the areas of the two circles separately. Then the difference of these areas will be the area of the ring.

Or, multiply the sum of the diameters by their difference.

and this product again by .7854, and it will give the area required.

EXAMPLES.

1. The diameters of the two circles are, A B 20, and D C 12 yards; required the area of the ring.




Here $400 \times .7854 = 314.16$, area of the outer circle.

$144 \times .7854 = 113.0976$, area of the inner circle.

And $314.16 - 113.0976 = 201.0624$ yards, area of the ring.

Or, $20 + 12 = 32$, sum of diameters.
$20 - 12 = 8$, difference of diameters.

And $32 \times 8 \times .7854 = 201.0624$ yards, area as before.

2. What is the area of a circular ring, the diameters of the concentric circles being 20 and 30 feet? *Ans.* 392.7 ft.

3. The diameters of two concentric circles are 8 and 12 yards; what is the area of the ring contained between their circumferences? *Ans.* 62.832 yds.

4. The diameters of two circles are $21\frac{3}{4}$ and $9\frac{1}{2}$ feet; required the area of the ring. *Ans.* 300.6609 ft.

5. The diameter of the inner circle is 6, and the outer 10 chains; what is the area of the ring? *Ans.* 50.2656 chs.

6. The area of the outer circle contains 100 acres, and the diameter of the inner is equal to $\frac{2}{3}$ of the diameter of the greater; what is the area of the ring? *Ans.* 555.55 chs.

## PROBLEM V.

The diameter or circumference of a circle being given, to find the side of an equivalent square.

RULE.—Multiply the diameter by .8862, or the circumference by .2821, the product in either case will be the side of an equivalent square.

EXAMPLES.

1. The diameter of a circle is 200 yards; what is the side of a square of equal area?

Here $200 \times .8862 = 177.24$ yards.

2. The circumference of a circle is 316 yards; what is he side of a square of equal area?

Here $316. \times .2821 = 89.1436$ yards.

3. The diameter of a circle is 1142 feet; what is the side of a square of equal area? *Ans.* 1012.0104 ft.

4. The circumference of a circle is 18.8 chains; what is the side of a square of equal area? *Ans.* 5.3034 chs.

5. The diameter of a circular fish-pond is 22 perches; what would be the side of a square fish-pond of an equal area? *Ans.* 19.4964 p.

6. The circumference of a circular walk is 64 rods; what is the side of a square containing the same area? *Ans.* 18.0544 r.

PROBLEM VI.

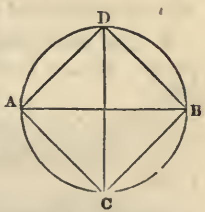
The diameter or circumference of a circle being given, to find the side of the inscribed square.

RULE.—Multiply the diameter by .7071, or the circumference by .2251, and the product in either case will be the side of the inscribed square.

EXAMPLES.

1. The diameter, A B, of a circle is 614 feet; what is the side, A C, of the inscribed square?

Here $614 \times .7071 = 434.1594$ feet $=$ A C.




2. The circumference of a circle is 1804 feet; what is the side of the inscribed square? *Ans.* 406.0804 ft.

3. The diameter of a circle is 239 feet; what is the side of the inscribed square? *Ans.* 168.9969 ft.

4. The circumference of a circle is 98 chains; what is the side of the inscribed square? *Ans.* 22.0598 chs.

5. The diameter of a circle is 65 rods; what is the side of the inscribed square? *Ans.* 45.9615 r.

6. The circumference of a circular walk is 721 perches; what is the side of an inscribed square?

*Ans.* 162.2971 p.

## PROBLEM VII.

To find the diameter of a circle equal in area to any given superficies.

Rule.—Divide the area by .7854, and the square root of the quotient will be the diameter.

### EXAMPLES.

1. The length and breadth of a rectangle are 24 and 16 chains; what is the diameter of a circle which contains the same area?

Here $24 \times 16 = 384$, the area of the rectangle.

Then $\sqrt{(384 \div .7854)} = \sqrt{488.9228} = 22.1116$ chains, the diameter.

2. The side of a square is 16 perches; what is the diameter of a circle containing the same area?

*Ans.* 18.054 p.

3. The base and perpendicular of a right-angled triangle are 16 and 20 feet; what will be the diameter of a circle which contains the same area? *Ans.* 14.2729 ft.

4. The three sides of a scalene triangle are 14, 18, and 24 yards; what is the diameter of a circle containing the sam area? *Ans.* 12.6267 yds.

5. The three sides of a triangle are 18, 20, and 26 feet; what is the diameter of a circle containing three times as much? *Ans.* 26.1919 ft.

6. The length and breadth of a parallelogram are 32 and 18 yards; what is the diameter of a circle that contains the same area? *Ans.* 27.0810 yds.

## PROBLEM VIII.

The diameter of a circle being given, to find another containing a proportionate quantity.

RULE.—Multiply the square of the given diameter by the given proportion, and the square root of the product will be the diameter required.

### EXAMPLES.

1. The diameter of a circle is 24 chains; what is the diameter of one containing one-fourth of the area?

Here $\sqrt{(24^2 \times \frac{1}{4})} = \sqrt{(576 \times \frac{1}{4})} = \sqrt{144} = 12$ chains.

2. The diameter of a circle is 16 perches; what is the diameter of one containing nine times as much?

*Ans.* 48 p.

3. The diameter of a circle is 36 yards; what is the diameter of one containing four times as much?

*Ans.* 72 yds.

4. The diameter of a circle is 81 feet; what is the diameter of one containing five times as much?

*Ans.* 181.1215 ft.

5. A gentleman has a circular grass-plat in his yard, the diameter of which is 25 yards; required the length of the string that would describe a circle to contain sixteen times as much.

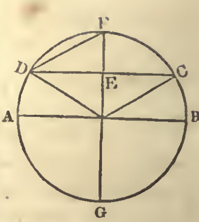
*Ans.* 50 yds.

6. The diameter of a circle is 9 rods; what is the diameter of one containing six times as much? *Ans.* 22.0454 r.

## PROBLEM IX.

To find the length of any arc of a circle.

RULE. 1.—From eight times the chord of half the arc, subtract the chord of the whole arc, and one-third of the remainder will be the length of the arc nearly.




The chord of the whole arc, or simply the chord, is a right line which joins the extremities of an arc. Thus in the figure, D C is the chord of the whole arc, D F C; D F the chord of half the arc, D F C; and D E, or E C, half the chord of the arc D F C. The height of an arc, called its versed sine, is that part of the diameter contained between the middle of the chord and the arc. Thus, in the case of the arc D F C, the versed sine, or the height of the arc, is the line E F.

The chord of half the arc may be found by adding together the square of half the chord and the square of the versed sine, and extracting the square root of the sum.

Or, by taking the square root of the product of the diameter and the versed sine.

Half the chord of the whole arc may be found by subtracting the versed sine from the diameter, multiplying the remainder by the versed sine, and taking the square root of the product. By doubling this we get the chord of the whole arc.

To find the versed sine, or height of the arc, subtract the square of the chord from the square of the diameter, and extract the square root of the remainder; subtract this root from the diameter, and one-half of the remainder will be the versed sine.

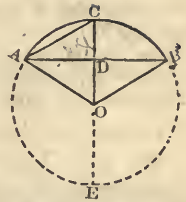
Or, from the square of the chord of half the arc, subtract the square of half the chord of the arc, and the square root of the remainder will give the versed sine.

Again, to obtain the versed sine, divide the square of the chord of half the arc by the diameter.

The diameter may be known by adding together the square of the versed sine and the square of half the chord of the arc, and dividing the sum by the versed sine. The diameter may likewise be obtained by dividing the square of the chord of half the arc by the versed sine.

It may here likewise be observed, as another rule for the same purpose, that if the number of degrees in the arc be multiplied by radius, and that product again by .01745, the result will give the length of the arc nearly.

EXAMPLES.




1. The chord A B of the whole arc A C B, is 48.74 feet, and the chord A C of half the arc 30.25 feet; what is the length of the arc?

Here $[(30.25 \times 8) - 48.74)] \div 3 = (242 - 48.74) \div 3 = 193.26 \div 3 =$ 64.42 ft. the length of the arc A C B nearly.

2. If the chord A B, of the arc A C B, be 30 yards, and the versed sine, or height C D, 8 yards, what is the length of the arc?

Here $\sqrt{(15^2 + 8^2)} = \sqrt{(225 + 64)} = \sqrt{289} = 17 =$ A C, the chord of half the arc.

And $[(17 \times 8) - 30)] \div 3 = (136 - 30) \div 3 = 106 \div 3 = 35\frac{1}{3}$ yards, the length of the arc nearly.

3. If the versed sine, or height of half the arc, be 4 feet, and the diameter of the circle 30 feet, what is the length of the arc?

Here $\sqrt{(30 \times 4)} = \sqrt{120} = 10.95445 =$ A C, the chord of half the arc.

And $\sqrt{[(30 - 4) \times 4]} \times 2 = \sqrt{(26 \times 4)} \times 2 = \sqrt{(104)} \times 2 = 10.19803 \times 2 = 20.39606 =$ A B, the chord of the whole arc.

Then $[(10.95445 \times 8) - 20.39606] \div 3 = (87.6356 - 20.39606) \div 3 = 67.23954 \div 3 = 22.41318$ feet, the length of the arc nearly.

4. Required the length of an arc of 30 degrees, the radius of the circle being 14 feet.

$(30 \times 14 \times .01745) = 7.329$ feet, the length of the arc.

5. The chord of the whole arc is $50\frac{4}{5}$ yards, and the chord of half the arc is $33\frac{1}{5}$ yards; required the length of the arc.

*Ans.* 71.6 yards.

6. The length of the chord of the whole arc is $36\frac{3}{4}$ feet, and the length of the chord of half the arc is $23\frac{1}{5}$ feet; what is the length of the arc? *Ans.* 49.6166 feet.

7. The chord of the whole arc is $48\frac{1}{2}$ feet, and its versed sine, or height of the arc, $18\frac{1}{4}$ feet; what is the length of the arc? *Ans.* 64.7667 feet.

8. If the versed sine or height of the arc be 2 yards, and the diameter of the circle 36 yards; what is the length of the arc? *Ans.* 17.1299 yards.

9. If the chord of the whole arc be 16 chains, and the radius of the circle 10 chains; what is the length of the arc? *Ans.* 18.518 chains.

10. The chord of half the arc is 10 perches, and the diameter of the circle $16\frac{2}{3}$ perches; what is the length of the arc? *Ans.* $21\frac{1}{3}$ perches.

11. Required the length of an arc of $57° 17' 44\frac{1}{2}''$, the diameter of the circle being 50 feet.

*Ans.* 25 feet, which equals the radius.

12. Required the length of a degree of a great circle of the earth, supposing its circumference to be 25000 miles. *Ans.* $69\frac{1}{2}$ miles nearly.

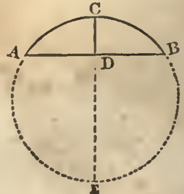
RULE 2.—Let $d$ = the diameter C E of the circle, and $v$ = the versed sine or height C D of half the arc, then will the length of the arc be expressed by the following series:

$$\text{Arc ACB} = 2\sqrt{dv} + \frac{v}{2\times 3\,d}\,A + \frac{3^2\,v}{4\times 5\,d}\,B + \frac{5^2\,v}{6\times 7\,d}\,C + \frac{7^2\,v}{8\times 9\,d}\,D, \text{ \&c.}$$

Where A, B, C, &c. represent the terms immediately preceding those in which they first occur.

Which rule is to be used, by substituting in the above series the numerical values of the given parts, in the place of the letters by which they are denoted, and then finding the sum of such a number of its terms as may be thought sufficient for determining the length of the arc, to a degree of accuracy required.

EXAMPLES.




1. Required the length of the arc, A C B, whose versed sine or height, D C, is 5 feet, and the diameter, C E, of the circle 25 feet.

Here $d = 25$, and $v = 5$.

Then $2\sqrt{dv} = 2\sqrt{(25 \times 5)} = 2\sqrt{125} = 22.3606798 = A$

$$\frac{v}{2 \times 3\,d} \times A = \frac{5}{6 \times 25} \times 22.3606798 = .7453559 = B$$

$$\frac{3^2\,v}{4 \times 5\,d} \times B = \frac{9 \times 5}{20 \times 25} \times .7453559 = .0670820 = C$$

$$\frac{5^2\,v}{6 \times 7\,d} \times C = \frac{25 \times 5}{42 \times 25} \times 0670820 = .0079859 = D$$

$$\frac{7^2\,v}{8 \times 9\,d} \times D = \frac{49 \times 5}{72 \times 25} \times .0079859 = .0010869 = E$$

The sum $= 23.1821905$

feet, = the length of the arc, A C B.

2. Required the length of the arc, A C B, whose versed sine or height, D C, is 2 feet, and the diameter, C E, of the circle 52 feet. *Ans.* 20.5291 feet.

3. Required the length of the arc whose versed sine is 9 yards, and the diameter of the circle 100 yards.
*Ans.* 60.9385 yards.

4. It is required to find the length of the arc whose chord is 16 perches, and its height 4 perches.
*Ans.* 18.5459 perches.

5. It is required to find the length of the arc whose height is 6 inches, and the chord of half the arc 1 foot.
*Ans.* 2.0943 feet.

6. It is required to find the length of the arc whose chord is 48 chains, and the radius of the circle 25 chains.
*Ans.* 64.3501 chains.

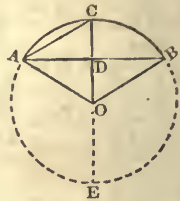
## PROBLEM X.

To find the area of a sector of a circle.

Rule 1.—Multiply the radius or half the diameter of the circle by half the length of the arc of the sector, as found by the last Problem, and the product will be the area.

To this we may add, that as 360° is to the number of degrees in the arc of the sector, so is the area of the circle, to the area of the sector.

EXAMPLES.




1 The chord, A B, of the whole arc, A C B, is 24 feet, and the chord, A C, of half the arc 13 feet; what is the area of the sector, O B C A O?

Here $\sqrt{(13^2-12^2)} = \sqrt{(169-144)} = \sqrt{25} = 5 =$ C D, the versed sine.

And $(13^2 \div 5) = 169 \div 5 = 33.8 =$ C E, the diameter.

$[(13 \times 8) - 24] \div 3 = (104 - 24) \div 3 = 80 \div 3 = 26\frac{2}{3}$ = the length of arc A C B.

Then $13\frac{1}{3} \times 16.9 = 225\frac{1}{3}$ feet, the area of the sector.

2. Required the area of the sector, the arc of which is 30°, and the diameter 3 yards.

Here $.7854 \times 9 = 7.0686 =$ the area of the circle.

As 360° : 30° :: 7.0686 : .58905 yards, the area of the sector.

3. The chord of the whole arc is 8 yards, and its height 3 yards; what is the area of the sector?

*Ans.* 22.2222 yards.

4. What is the area of a sector whose chord is $18\frac{3}{5}$ chains, and the diameter of the circle 20 chains?

*Ans.* 118.954 chains.

5. Required the area of the sector whose height is 4 perches, and the radius of the circle 8 perches.

*Ans.* 66.8581 perches.

6. Required the area of the sector whose arc is 17° 15′, and the diameter of the circle 19 feet.

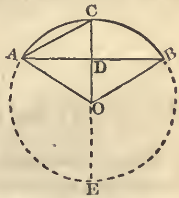
*Ans.* 13.5857 feet.

Rule 2.—Let $d$ = the diameter C E of the circle, and $v$ = the versed sine or height, C D, of half the arc, then will the area of the sector be expressed by the following series.

$$\text{Area OACBO} = \tfrac{1}{2} d \sqrt{dv} + \frac{v}{2 \times 3 d} \text{A} + \frac{3^2 v}{4 \times 5 d} \text{B} + \frac{5^2 v}{6 \times 7 d} \text{C} + \frac{7^2 v}{8 \times 9 d} \text{D}, \&c.$$

When A, B, C, &c., represent the terms immediately preceding those where they first occur. This Rule is to be used in the same manner as directed in regard to Rule 2, Problem IX. p. 96.

### EXAMPLES.




1. Required the area of the sector, O A C B O, whose versed sine, or height, C D, is 7 feet, and the diameter, C E, of the circle 28 feet.

Here $d=28$, and $v=7$.

Then $\frac{1}{2}d\sqrt{dv}=14\sqrt{(28\times 7)}=14\sqrt{196}=196.000000=A$

$$\frac{v}{2\times 3d}\times A=\frac{7}{6\times 28}\times 196 = 8.166666=B$$

$$\frac{3^2 v}{4\times 5d}\times B=\frac{9\times 7}{20\times 28}\times 8.166666= .918750=C$$

$$\frac{5^2 v}{6\times 7d}\times C=\frac{25\times 7}{42\times 28}\times .918750= .136718=D$$

$$\frac{7^2 v}{8\times 9d}\times D=\frac{49\times 7}{72\times 28}\times .136718= .023261=E$$

$$\frac{9^2 v}{10\times 11d}\times E=\frac{81\times 7}{110\times 28}\times .023261= .004282=F$$

$$\frac{11^2 v}{12\times 13d}\times F=\frac{121\times 7}{156\times 28}\times .004282= .000830=G$$

The sum $=205.250507$ feet = the area of the sector O A C B O.

2. Required the area of the sector O A C B O, whose versed sine, C D, is 2 yards, and the diameter, C E, of th circle 52 yards. *Ans.* 266.8787 yds.

3. Required the area of the sector, the radius of the circle being 10 feet, and the chord of the arc 12 feet.

*Ans.* 64.3501 ft.

4. Required the area of the sector of a circle whose diameter is 20 perches, and the chord of its arc 16 perches.

*Ans* 92.7295 p

5. Required the area of the sector of a circle whose versed sine is 9 feet, and the chord of half the arc 30 feet.

*Ans.* 1523.4632 ft.

6. Required the area of the sector of a circle whose chord is 48 feet, and versed sine 18 feet. *Ans.* 804.376 ft.

## PROBLEM XI.

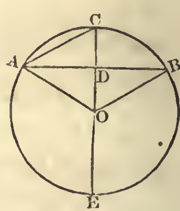
To find the area of a segment of a circle.

Rule 1.—Find the area of the sector, having the same arc as the segment, by the last problem.

Also find the area of the triangle formed by the chord of the segment, and the two radii of the sector, by Problem I. page 67.

The sum, or difference of these areas, according as the segment is greater or less than a semicircle, will be the area of the segment required.

### EXAMPLES.




1. The chord A B is 24 feet, and the versed sine or height C D, of half the arc A C B, is 5 feet; what is the area of the segment A B C A?

Here $\sqrt{(A D^2 + C D^2)} = \sqrt{(12^2 + 5^2)} = \sqrt{(144 + 25)} = \sqrt{169} = 13 =$ A C, the chord of half the arc.

Again, $(A C^2 \div C D) = (169 \div 5) = 33.8 =$ C E, the diameter; therefore $16.9 =$ C O, the radius.

Then $[(13 \times 8) - 24] \div 3 = 26\frac{2}{3}$, the length of the arc A C B.

And $(13\frac{1}{3} \times 16.9) = 225.3333$ feet, the area of the sector O A C B O.

But $(C O - C D) = (16.9 - 5) = 11.9 =$ perpendicular O D.

Whence $(11.9 \times 12) = 142.8$, area of the $\triangle$ A O B.

And, consequently, $(225.3333 - 142.8) = 82.5333$ feet area of the segment A B C A.

2. What is the area of a segment of a circle whose arc is 60°, and the diameter of the circle 10 feet?

Here $.7854 \times 100 = 78.54$, area of the whole circle.

Then, as $360° : 60° :: 78.54 : 13.09$, area of the sector O A C B O.

And since the chord A B (to an arc of 60°) is = radius O A or O C, which = 5, then by Problem II., page 82, the area of the $\triangle$ A O B = 10.8253.

Whence $(13.09 - 10.8253) = 2.2647$ feet, the area of the segment required.

3. Required the area of the segment of a circle, whose versed sine, or height, of half the arc is 5 yards, and the diameter of the circle 20 yards. *Ans.* 61.1645 yds.

4. Required the area of the segment of a circle, whose chord is 16 chains, and diameter of the circle $16\frac{2}{3}$ chains. *Ans.* 70.2222 chs.

5. What is the area of the segment of a circle, whose arc is a quadrant, the diameter being 24 perches? *Ans.* 41.0976 p.

6. Required the area of a segment of a circle, whose chord is 18.9 feet, and height 2.4 feet. *Ans.* 30.601 ft.

Rule 2.—Let $d$ equal the diameter C E of the circle, and $v$ equal the versed sine C D, of half the arc, or height of the segment; then will the area of the segment be expressed by the following series.

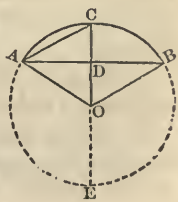
Area of the segment A C B A =

$$\frac{4v\sqrt{dv}}{3} - \frac{3v}{2 \times 5d}A - \frac{5v}{4 \times 7d}B - \frac{3 \times 7v}{6 \times 9d}C - \frac{5 \times 9v}{8 \times 11d}D.$$

Where A, B, C, &c., are the first, second, third, &c., terms of the series.

This rule is to be used in a manner similar to that directed in regard to Rule 2d, Problem IX., page 96.

EXAMPLES.




1. Required the area of the segment A B C A of a circle, whose chord A B is 16 feet, and the diameter C E 20 feet.

Here $\sqrt{(CE^2 - AB^2)} = \sqrt{(20^2 - 16^2)} = \sqrt{(400 - 256)} = \sqrt{144} = 12$; then $(20 - 12) \div 2 = 4 = $ C D, the versed sine.

$$\frac{4v\sqrt{dv}}{3} = \frac{16\sqrt{(20\times 4)}}{3} = \frac{16\sqrt{80}}{3} = \qquad +\underline{47.702783} = A$$

$$-\frac{3v}{2\times 5d}\times A = -\frac{3\times 4}{10\times 20}\times +47.702783 = -2.862167 = B$$

$$-\frac{5v}{4\times 7d}\times B = -\frac{5\times 4}{28\times 20}\times -2.862167 = -.102220 = C$$

$$-\frac{3\times 7v}{6\times 9d}\times C = -\frac{21\times 4}{54\times 20}\times -.102220 = -.007950 = D$$

$$-\frac{5\times 9v}{8\times 11d}\times D = -\frac{45\times 4}{88\times 20}\times -.007950 = -.000813 = E$$

$$-\frac{7\times 11v}{10\times 13d}\times E = -\frac{77\times 4}{130\times 20}\times -.000813 = -\underline{.000096} = F$$

The sum of B C D E F $= -\underline{2.973246}$

The difference $=$ 44.729537, ft.

$=$ the area of the segment A B C A.

2. Required the area of the segment of a circle whose height is 24 feet, and the radius 26 feet. *Ans.* 957.9609 ft.

3. Required the area of the segment of a circle whose versed sine is 3 yards, and the diameter 60 yards.

*Ans.* 52.8533 yds.

4. Required the area of the segment of a circle whose height is 4 feet, and the radius 50 feet.

*Ans.* 105.3773 ft.

5. Required the area of the segment of a circle whose chord is 20 chains, and the diameter 52 chains.

*Ans.* 26.8787 chs.

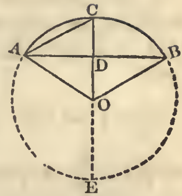
6. Required the area of the segment of a circle whose height is 2 feet, and chord 12 feet. *Ans.* 16.35 ft.

RULE 3.—1. From seven times the diameter subtract five times the versed sine; multiply the remainder by seven times the versed sine, and extract the square root of the product.

2. Multiply the diameter by the versed sine, and extract the square root of the product.

3. To the first root add four thirds of the second, and multiply the sum by four twenty-fifths of the versed sine, and the product will be the area of the segment nearly.

EXAMPLES.




1. Required the area of the segment, A B C A, of a circle whose versed sine, or height, C D, is 4 feet, and the diameter, C E, 20 feet.

Here $\sqrt{\{[(20\times7)-(4\times5)]\times(4\times7)\}}$ $=\sqrt{[(140-20)\times28]}=\sqrt{(120\times28)}$ $=\sqrt{3360}=57.96550$ the first root.

And $\sqrt{(20\times4)}=\sqrt{80}=8.94427$, the second root.

Then $57.96550+(8.94427\times\frac{4}{3})=(57.96550+11.92569)$ $=69.89119$.

Whence $69.89119\times(4\times\frac{4}{25})=(69.89119\times.64)=44.73036$ feet, the area of the segment A B C A.

2. What is the area of a segment of a circle whose versed sine, or height, is 4 feet, and the diameter of the circle is 40 feet? *Ans.* 65.4005 feet.

3. Required the area of the segment of a circle whose versed sine is 4 yards, and the diameter 100 yards. *Ans.* 105.3773 yards.

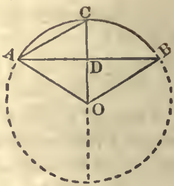
4. Required the area of the segment of a circle whose height is 3 yards, and the diameter 60 yards. *Ans.* 52.8533 yards.

5. What is the area of a segment of a circle whose height, or versed sine, is 5 feet, and the diameter of the circle 25 feet? *Ans.* 69.8911 ft.

6. What is the area of the segment of a circle whose height, or versed sine, is 3 feet, and the diameter of the circle 8 feet? *Ans.* 17.2198 ft.

Rule 4.—To two-thirds of the product of the chord and versed sine of the segment, add the cube of the versed sine divided by twice the chord, and the sum will give the area of the segment nearly.

EXAMPLES.




1. What is the area of the segment, A B C A, of a circle whose versed sine, C D, is 4 feet, and the chord, A B, is 16 feet?

Here $[(16\times 4)\times \frac{2}{3}]+[4^3\div(16\times 2)]$

$= (64\times\frac{2}{3}) + (64\div 32) = (\frac{128}{3}+2)$

$= (42\frac{2}{3}+2) = 44\frac{2}{3}$ feet, the area of the segment A B C A.

2. What is the area of the segment of a circle whose versed sine is 2 feet, and the chord 20 feet?

*Ans.* $26.8\frac{2}{3}$ ft.

3. Required the area of the segment of a circle whose height is 2 feet, and the chord 12 feet. *Ans.* $16\frac{1}{3}$ ft.

4. Required the area of the segment of a circle whose versed sine is 15 feet, and the chord 40 feet.

*Ans.* 442.1875 ft.

5. Required the area of the segment of a circle whose versed sine is 2 feet, and the chord 7 feet. *Ans.* $9\frac{10}{21}$ ft.

6. Required the area of the segment of a circle whose versed sine is 18 feet, and the chord 48 feet.

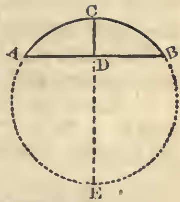
*Ans.* $636\frac{3}{4}$ ft.

Rule 5.—Divide the versed sine, or height of the segment by the diameter of the circle, and find the quotient in the table of versed sines, p. 295.

Then multiply the tabular area on the right hand of the versed sine so found, (which is the tabular segment,) by the square of the diameter, and the product will be the area.

When the quotient arising from dividing the versed sin by the diameter has a remainder, or fraction, after the third place of decimals, subtract the tabular area, answering to the first three figures, from the next following area; then if the remainder be multiplied by the fractional part, and the result be added to the first area, it will give the tabular area for the whole quotient; which must be multiplied by the square of the diameter, as before.

EXAMPLES.




1. What is the area of the segment of a circle whose versed sine, C D, is 2 yards, and the diameter of the circle, C E, 52 yards?

Here $(2 \div 52) = .038\frac{6}{13}$, the tabular versed sine.

And .009763 the tab. segment to .038.
.010148 the next do. .039.
.000385 difference.

Then $.000385 \times \frac{6}{13} = .000178$

.009763 tab. segment to .038.
.009941 tab. segment to $.038\frac{6}{13}$.

Now $(.009941 \times 52^2) = (.009941 \times 2704) = 26.880464$ yards, the area of the segment.

2. What is the area of the segment of a circle whose height is 10 feet, and the diameter of the circle 50 feet?

Here $(10 \div 50) = .2$ the tabular versed sine.

And .111823, the tabular segment to .2.

Now $(.111823 \times 50^2) = (.111823 \times 2500) = 279.5575$ feet, the area required.

3. What is the area of the segment of a circle whose versed sine is 3 feet, and the diameter of the circle 8 feet?

*Ans.* 17.2168 ft.

4. What is the area of the segment of a circle whose height is 15 yards, and the diameter of the circle 75 yards?

*Ans.* 629.0043 yds.

5. What is the area of the segment of a circle whose height is 6 feet, and the diameter of the circle 21 feet?

*Ans.* 81.6582 ft.

6. What is the area of the segment of a circle whose versed sine is 7 feet, and the diameter of the circle 38 feet?

*Ans.* 143.5104 ft.

## PROBLEM XII.

To find the area of a circular zone, or the space included between two parallel chords and their intercepted arcs.

RULE. Find the area of that part of the zone A B C D, which forms a trapezoid, by Prob. I. p. 80, and the area of the small segment B $n$ C B by rule 5, in the last Problem.

Then add the area of the trapezoid to twice the area of the segment, and it will give the area of the zone.

1. Let $d$ = the diameter E F; C, $c$ = the two parallel chords A B, D C; $v$ = the versed sine $m\,n$, and $b$ = G H, the breadth of the zone, we shall have

$$EF = \sqrt{\left\{b^2 + \frac{C^2 + c^2}{2} + \left(\frac{C^2 - c^2}{4b}\right)^2\right\}}.$$

And,

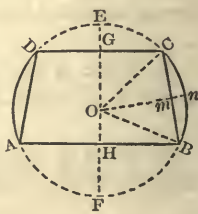
$$mn = \frac{1}{2}d - \frac{1}{2}\sqrt{\left\{\left(\frac{C + c}{2}\right)^2 + \left(\frac{C^2 - c^2}{4b}\right)^2\right\}}.$$

2. When the two parallel chords are equal to each other, the diameter $d$ or E F will be $= \sqrt{(b^2 + c^2)}$; and the versed sine $mn = \frac{1}{2}(d - c)$.

And when one of the parallel chords is the diameter, the versed sine $mn = \frac{1}{2}d - \frac{1}{2}\sqrt{\left\{b^2 + \left(\frac{d + c}{2}\right)^2\right\}}$,

which expressions will be found of considerable use in facilitating the computation of the segment B $n$ C B.

EXAMPLES.




1. The greater chord A B is 8 yards, the less D C 6 yards, and their perpendicular distance G H is 7 yards; required the area of the zone.

Here A B + D C = 8 + 6 = 14, and G H = 7.

Whence $(14 \times 7) \div 2 = 49$, the area of the trapezoid A B C D

$$\text{Also} \quad EF = \sqrt{\left\{7^2 + \frac{8^2 + 6^2}{2} + \left(\frac{8^2 - 6^2}{4 \times 7}\right)^2\right\}} =$$

$\sqrt{(49 + 50 + 1)} = \sqrt{100} = 10$ the diameter.

$$\text{And} \quad mn = \frac{10}{2} - \frac{1}{2}\sqrt{\left\{\left(\frac{8 + 6}{2}\right)^2 + \left(\frac{8^2 - 6^2}{4 \times 7}\right)^2\right\}} =$$

$5 - \frac{1}{2}\sqrt{50} = 5 - 3.535533 = 1.464467$, the versed sine.

Therefore $1.464467 \div 10 = .1464467$, the tabular versed sine.

Answering to which, when the work is performed as in the last problem, we shall have .071349.

Hence $.071349 \times 10^2 = .071349 \times 100 =$

7.1349, the area of the segment B $n$ C B.

$7.1349 \times 2 = 14.2698$, twice do.

49.0000, area of the trapezoid A B C D.

63.2698 yards, area of the zone.

2. One of the parallel chords of a circular zone is 48 feet, and the other 30 feet, and its breadth 13 feet; what is the area of the zone? *Ans.* 534.19 ft.

3. The greater chord of a circular zone is 16 yards, and the less chord 12 yards, and their perpendicular distance 2 yards; what is the area of the zone? *Ans.* 28.379 yds.

4. Supposing the greater chord of a circular zone to be 20 feet, and the less 15 feet, and their distance $17\frac{1}{2}$ feet; what is the area of the zone? *Ans.* 395.4362 ft.

5. Required the area of a circular zone, each of whose parallel chords are 50 feet, and their perpendicular distance 30 feet. *Ans.* 1668.7093 ft.

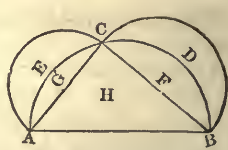
6. It is required to find the area of a circular zone, the greater chord of which, being equal to the diameter of the circle, is 40 feet, and the less 20 feet. *Ans.* 592.086 ft.

## PROBLEM XIII.

To find the area of a lune, or the space included between the intersecting arcs of the two eccentric circles.

Rule. Find the areas of the two segments, from which the lune is formed by Problem XI., rule 5, page 104, and their difference will be the area of the lune.

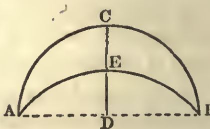
If semicircles be described on the three sides of a right-angled triangle as diameters, the two lunes formed on the base and perpendicular, taken together, will be equal to the area of the triangle.




If A B C, or H, be a right-angled triangle, and semicircles be described on the three sides as diameters, then will the triangle H be equal to the two lunes D and E taken together.

For, if from the greater semicircle A B C there be taken the two segments F and G, there will remain the triangle H; and if the same segments be taken from the other two semicircles, there will remain the lunes D and E: hence, since the greater semicircle is equal to the sum of the other two, the triangle H must be equal to the sum of the lunes D and E.

EXAMPLES.




1. The length of the chord A B is 40 feet, the height D C 10 feet, and D E 4 feet; required the area of the lune.

Here $(A\,D^2 + C\;D^2) \div C\;D = (20^2 + 10^2) \div 10 = (400 + 100) \div 10 = 500 \div 10 = 50$, the diameter belonging to the circle of which A C B is a part.

Again $(A\,D^2 + D\,E^2) \div D\,E = (20^2 + 4^2) \div 4 = (400 + 16) \div 4 = 416 \div 4 = 104$, the diameter belonging to the circle of which A E B is a part.

Whence $10 \div 50 = .2$ the first tabular versed sine.

Answering to which is .111823, the first tabular segment.

Therefore $.111823 \times 50^2 = .111823 \times 2500 = 279.5575$, the area of the segment A B C A.

Also $4 \div 104 = .038\frac{6}{13}$, the second tabular versed sine.

Answering to which is .009941, the second tabular segment.

Consequently, $.009941 \times 104^2 = .009941 \times 10816 = 107.521856$, the area of the segment A E B A.

Whence $279.5575 - 107.521856 = 172.035644$ feet, area of the lune A C B E A.

2. The chord is 48 feet, and the height of the segments 18 and 12 feet; required the area of the lune.

*Ans.* 233.8122 ft.

3. The chord is 40 feet, and the height of the segments 20 and 4 feet; required the area of the lune.

*Ans.* 520.7965 ft.

4. Supposing the length of the chord to be 96 feet, and the height of the segments to be 36 and 14 feet; what is the area of the lune? *Ans.* 1634.4350 ft.

5. The length of the chord is 50 feet, and the heights of the segments 18 and 15 feet; required the area of the lune.

*Ans.* 123.8785 ft.

6. The length of the chord is 40 feet, and the heights of the segments 10 and 8 feet; what is the area of the lune?

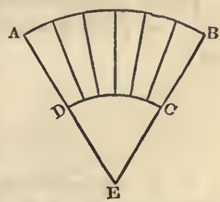
*Ans.* 59.5485 ft.

## PROBLEM XIV.

To find the area of a part of a ring, or of the segment of a sector.

RULE.—Multiply half the sum of the bounding arcs by their distance asunder, and the product will give the area.

### EXAMPLES.




1. Let A B be 50 inches, and D C 30 inches, and the distance D A 10 inches; what is the area of the space A B C D?

Here $[(50 + 30) \div 2] \times 10 = (80 \div 2) \times 10 = 40 \times 10 = 400$ inches.

2. Let A B = 60 inches, D C = 40 inches, and D A = 2; required the area of the space A B C D. *Ans.* 100 in.

3. Let A B = 25 feet, D C = 15 feet, and D A = 6 feet required the area of the segment of the sector.

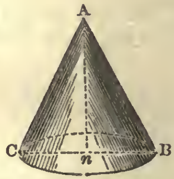
*Ans.* 120 ft.

# CONIC SECTIONS.

## DEFINITIONS.

§ **15.** 1. The conic sections are certain plane figures formed by the cutting of a cone, which are of great use in some of the higher branches of mathematics.

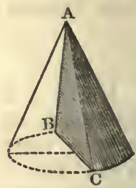
2. A *cone* is a solid described by the revolution of a right-angled triangle about one of its legs, which remains fixed; as A B C.



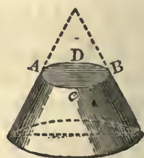

3. The *axis of the cone* is the right line about which the triangle revolves.

4. The base of a cone is the circle which is described by the revolving leg of the triangle; and its altitude is a perpendicular drawn from the vertex to the base; as A $n$.

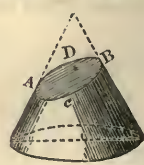
5. If a cone be cut through the vertex by a plane, perpendicular or oblique to that of the base, the section will be a triangle; as A B C.

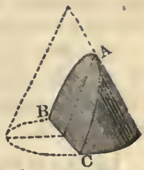



6. If a cone be cut into two parts, by a plane parallel to the base, the section will be a circle, as A C B D.

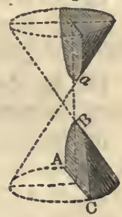



7. If a cone be cut by a plane which passes obliquely through its two slant sides, the section will be an ellipse, as A C B D.





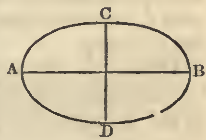

8. If a cone be cut by a plane, which is parallel to either of its slant sides, the section will be a parabola; as A B C A.



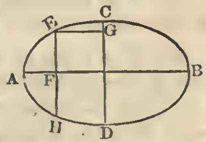

9. If a cone be cut into two parts, by a plane, which, being continued, would meet the opposite cone, the section is called an hyperbola; as A B C.

The two opposite cones, in this definition, are supposed to be generated together, by the revolution of the same line.

It may here also be observed, that all figures which can possibly be formed by the cutting of a cone, are mentioned in these definitions, and are the five following: viz., a *triangle*, a *circle*, an *ellipsis*, a *parabola*, and an *hyperbola;* but the last three only are usually called the *conic sections.*




10. If two lines be drawn through the centre of an ellipse, perpendicular to each other, and terminated both ways by the circumference, the longest of them, A B, is called the transverse diameter, or axis, and the shortest, C D, the conjugate.



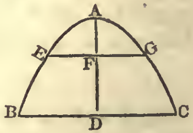

11. An ordinate of an ellipse is a right line E F, or E G, drawn from any point E, in the curve, perpendicular to either of the diameters.

12. An abscissa is that part, A F, or C G, of the diameter which is contained between either of the extremities of that diameter and the ordinate.

An abscissa may be more generally considered as any part of the diameter or axis of a curve comprised between any fixed point, from which all the abscissas are supposed to take their origin, and another line called the ordinate, drawn so as to make a given angle with the former, and terminated in the curve; but when not otherwise specified, they are commonly taken as above. The abscissa and its correspond-

ing ordinates, when considered together, are also frequently called co-ordinates.

13. The axis of a parabola, B A C, is a right line, A D, drawn from the vertex, so as to divide the figure into two equal parts.




14. An ordinate of a parabola is a right line E F, drawn from any point in the curve, perpendicular to the axis.

15. An abscissa is that part of the axis, A F, which is contained between the vertex of the curve and the ordinate.

16. The transverse diameter of an hyperbola is that part of the axis which is intercepted between the two opposite cones, as $a$ B, in the figure accompanying definition ninth.

17. The conjugate diameter is a line drawn through the centre, perpendicular to the transverse.

18. An ordinate of an hyperbola is a line drawn from any point in the curve, perpendicular to either of the diameters; and an abscissa is that part of the diameter which is contained between either of the extremities of that diameter and the ordinate.

Hence, in the ellipse and hyperbola, every ordinate has two abscissas, but in the parabola only one, the other vertex of the diameter being at an infinite distance.

19. The parameter of any diameter is a third proportional to that diameter and its conjugate.

20. The focus of a conic section is the point in the axis where the ordinate is equal to half the parameter, as F, in the figure accompanying definition eleventh, where E F is equal to the semi-parameter of the section.

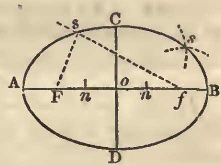
The ellipse and hyperbola have each two foci, as F, $f$: in the figure accompanying the following problem, the parabola has but one.

21. The vertices of a conic section are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section. Hence the ellipse and the opposite hyperbolas have each two vertices, as A, B, and B, $a$, in the figures accompanying definitions seventh and ninth; but the parabola only one, as A, in the figure accompanying definition eighth.

# THE ELLIPSE.

## PROBLEM I.

§ **16.** To describe an ellipse, the transverse and conjugate diameters being given.



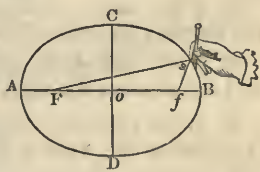

*Construction.* 1. Draw the transverse and conjugate diameters, A B, C D, bisecting each other perpendicularly in the centre *o*.

2. With the radius A *o*, and centre C, describe an arc cutting A B in F, *f*; and these two points will be the foci of the ellipse.

3. Take any number of points *n*, *n*, &c., in the transverse diameter A B, and with the radii A *n*, *n* B, and centres F, *f*, describe arcs intersecting each other in *s*, *s*, &c.

4. Through the points *s*, *s*, &c., draw the curve A *s* C *s* B D, and it will be the circumference of the ellipse required.

It is a well-known property of the ellipse, that the sum of two lines drawn from the foci, to meet in any point in the curve, is equal to the transverse diameter; and from this the truth of the construction is evident.




From the same principle is derived another method of describing an ellipse, by means of a string and two pins.

Having found the foci F, *f*, as before, take a thread of the length of the transverse diameter, and fasten its ends with two pins in the points F, *f*; then stretch the thread F *s f* to its greatest extent, and it will reach to the point *s* in the curve; and by moving a pencil round within the thread, keeping it always stretched, it will trace out the curve required.

## PROBLEM II.

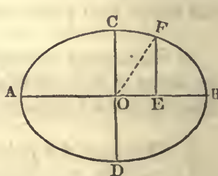
In an ellipse, any three of the four following terms being given, viz., the transverse and conjugate diameters, an ordinate and its abscissa, to find the fourth.

CASE 1.—When the transverse, conjugate, and abscissa are given, to find the ordinate.

RULE.—As the transverse diameter is to the conjugate, so is the square root of the product of the two abscissas, to the ordinate which divides them.

It may be remarked that if but one abscissa is given, by subtracting it from the transverse diameter, we obtain the other.

### EXAMPLES.




1. In the ellipse A D B C, the transverse diameter A B is 50, the conjugate diameter C D is 30, and the abscissa B E 18; what is the length of the ordinate E F?

Here $AB = 50$, $CD = 30$, $BE = 18$, and $AE = 50 - 18 = 32$.

Whence $50 : 30 :: \sqrt{(18 \times 32)} : EF$. Or, $EF = \frac{30}{50}\sqrt{576} = \frac{3}{5} \times 24 = \frac{72}{5} = 14.4$, the ordinate required.

2. If the transverse diameter be 40, the conjugate 30, and the abscissa 24, what is the ordinate? *Ans.* 14.6969.

3. If the transverse diameter be 120, the conjugate 40, and the abscissa 24, what is the ordinate? *Ans.* 16.

4. If the transverse diameter be 35, the conjugate 25, and he abscissa 28, what is the ordinate? *Ans.* 10.

CASE 2.—When the transverse, conjugate, and ordinate are given, to find the abscissa.

RULE.—As the conjugate diameter is to the transverse, so is the square root of the difference of the squares of the ordinate and semi-conjugate to the distance between the ordinate and centre.

And this distance being added to, and subtracted from, the semi-transverse, will give the two abscissas required.

EXAMPLES.

1. The transverse diameter A B is 60, the conjugate diameter C D 40, and the ordinate F E 12; what is the length of each of the two abscissas B E and A E?

Here A B $= 60$, C D $= 40$, and F E $= 12$.

Whence $40 : 60 :: \sqrt{(20^2 - 12^2)} :$ O E.

Or O E $= \frac{60}{40}\sqrt{(400 - 144)} = \frac{3}{2}\sqrt{256} = \frac{3}{2} \times 16 = \frac{48}{2} = 24$.

And $30 - 24 = 6 =$ B E, and $30 + 24 = 54 =$ A E.

2. What are the two abscissas to the ordinate 10, the diameters being 35 and 25? *Ans.* 7 and 28.

3. What are the two abscissas to the ordinate 16, the diameters being 120 and 40? *Ans.* 24 and 96.

4. What are the two abscissas to the ordinate 14.4, the diameters being 50 and 30? *Ans.* 18 and 32.

CASE 3.—When the conjugate, ordinate, and abscissa are given, to find the transverse diameter.

RULE.—To, or from the semi-conjugate, according as the less or greater abscissa is used, add or subtract the square root of the difference of the squares of the ordinate and semi-conjugate.

Then, as the square of the ordinate is to the product of the conjugate and abscissa, so is the sum or difference above found to the transverse diameter required.

EXAMPLES.

1. The conjugate diameter C D is 60, the ordinate E F 24, and the less abscissa B E 36; required the transverse diameter A B.

Here O C $= 30$, E F $= 24$, and B E $= 36$.

Whence $30 + \sqrt{(30^2 - 24^2)} = 30 + \sqrt{(900 - 576)} = 30 + \sqrt{324} = 30 + 18 = 48$.

And $24^2 : (60 \times 36) :: 48 : (60 \times 36 \times 48) \div 24^2 = 103680 \div 576 = 180$, the transverse diameter required.

2. The conjugate diameter is 60, the ordinate 18, and the less abscissa 24; required the transverse diameter.

*Ans.* 240.

3. The conjugate diameter is 40, the ordinate 16, and the greater abscissa 96; required the transverse diameter.

*Ans.* 120.

4. The conjugate diameter is 25, the ordinate 10, and the greater abscissa 28; required the transverse diameter.

*Ans.* 35.

Case 4. The transverse, ordinate, and abscissa being given, to find the conjugate diameter.

Rule.—As the square root of the product of the two abscissas is to the ordinate, so is the transverse diameter to the conjugate.

### EXAMPLES.

1. The transverse A B is 82, the ordinate E F 8, and the abscissa B E 32; required the conjugate diameter C D.

Here $AB = 82$, $EF = 8$, $BE = 32$, and $AE = 82 - 32 = 50$.

Hence $\sqrt{(32 \times 50)} : 8 :: 82 : (8 \times 82) \div \sqrt{(32 \times 50)} = 656 \div \sqrt{1600} = 656 \div 40 = 16.4$, the conjugate diameter required.

2. The transverse diameter is 35, the ordinate 10, and the abscissa 28; what is the conjugate? *Ans.* 25.

3. The transverse diameter is 120, the ordinate 16, and its abscissa 24; what is the conjugate? *Ans.* 40.

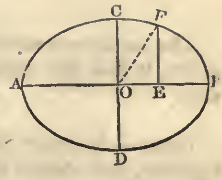
4. The transverse diameter is 50, the ordinate 14.4, and the abscissa 32; what is the conjugate? *Ans.* 30.

### PROBLEM III.

The transverse and conjugate diameters being given, to find the circumference.

Rule.—Multiply the square root of half the sum of the squares of the two diameters by 3.1416, and the product will be the circumference nearly.

EXAMPLES.




1. The transverse is 40, and the conjugate 30; required the circumference of the ellipse.

Here $\sqrt{[(40^2 + 30^2) \div 2]} = \sqrt{[(1600 + 900) \div 2]} = \sqrt{(2500 \div 2)} = \sqrt{1250} = 35.3553$.

Whence $35.3553 \times 3.1416 = 111.0722$ = the circumference.

2. The transverse diameter is 24, and the conjugate 20; required the circumference of the ellipse. *Ans.* 69.4001.

3. The transverse diameter is 60, and the conjugate 40; what is the circumference? *Ans.* 160.1907.

4. The transverse diameter is 24, and the conjugate 18; what is the circumference? *Ans.* 66.6433.

## PROBLEM IV.

The transverse and conjugate diameters being given, to find the area.

RULE.—Multiply the transverse diameter by the conjugate, and this product again by .7854, and the result will be the area.

EXAMPLES.

1. Required the area of an ellipse whose transverse diameter is 24 perches, and the conjugate 18 perches.

Here $24 \times 18 \times .7854 = 339.2928$ perches = 2 a. 0 r. 19.2928 p., the area required.

2. Required the area of an ellipse whose two diameters are 60 and 40 rods. *Ans.* 11 a. 3 r. 4.96 p.

3. Required the area of an ellipse whose two diameters are 40 and 36 chains. *Ans.* 113 a. 0 r. 15.616 p.

4. The transverse and conjugate diameters are 66 and 22 yards; what is the area? *Ans.* 1140.4 yds.

## PROBLEM V.

The transverse and conjugate diameters of an ellipse being given, to find the diameter of a circle containing the same area.

RULE.—Multiply the transverse and conjugate diameters together, and extract the square root of their product.

### EXAMPLES.

1. The transverse and conjugate diameters are 70 and 50 chains; what is the diameter of a circle containing the same area?

Here $\sqrt{(70 \times 50)} = \sqrt{3500} = 59.16079$ chains, the diameter.

2. The transverse and conjugate diameters are 36 and 24 chains; what is the diameter of a circle containing the same area? *Ans.* 29.3938 chs.

3. The transverse and conjugate diameters are 49 and 16 rods; what is the diameter of a circle containing the same area? *Ans.* 28 rods.

4. The transverse and conjugate diameters are 87 and 52 feet; what is the diameter of a circle containing the same area? *Ans.* 67.2606 ft.

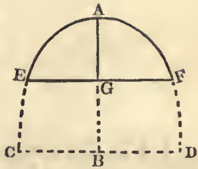
## PROBLEM VI.

To find the area of an elliptic segment whose base is parallel to either of the axes of the ellipse.

RULE.—Divide the height of the segment by that axis of the ellipse of which it is a part, and find in the table, p. 295, a circular segment whose versed sine is equal to the quotient.

Then, multiply the segment thus found and the two axes of the ellipse together, and the product will be the area required.

EXAMPLES.




1. Required the area of the elliptic segment whose height A G is 20 chains, and the axes of the ellipse, 2 A B and C D, 70 and 50 chains respectively.

Here $20 \div 70 = .285\frac{5}{7}$, the tabular versed sine. And the tabular segment belonging to this, as found by Prob. XI., Rule 5, p. 104, is .185166.

Whence $.185166 \times 70\ (2\,AB) \times 50\ (CD) = 648.081$ chains = 64 a. 3 r. 9.296 p., the area of the segment E A F.

2. What is the area of an elliptic segment cut off by an ordinate parallel to the transverse diameter whose height is 20 feet, the axes being 80 and 50 feet?

*Ans.* 1173.476 ft.

3. What is the area of an elliptic segment cut off by an ordinate parallel to the transverse diameter whose height is 10 chains, the axes being 35 and 25 chains?

*Ans.* 25 a. 2 r. 27.166 p.

4. What is the area of an elliptical segment cut off by an ordinate parallel to the conjugate diameter whose height is 10 feet, the axes of the ellipse being 35 and 25 feet?

*Ans.* 162.0202 ft.

5. What is the area of an elliptic segment cut off by an ordinate parallel to the transverse diameter whose height is 5 yards, the axes being 35 and 25 yards?

*Ans.* 97.8451 yds.

6. What is the area of an elliptic segment cut off by an ordinate parallel to the conjugate diameter, the axes of the ellipse being 60 and 40 feet, and the height of the segment 15 feet? *Ans.* 368.51 ft.

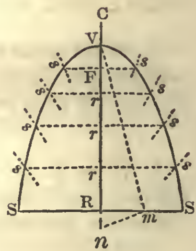
7. What is the area of an elliptic segment cut off by a double ordinate parallel to the conjugate axis, at the distance of 36 yards from the centre, the axes being 120 and 40 yards?

*Ans.* 536.7504 yds.

## THE PARABOLA.

### PROBLEM I.

§ **17.** To describe a parabola, any ordinate to the axis and its abscissa being given.




*Construction.* 1. Let V R and R S be the given abscissa and ordinate; bisect the latter in $m$, join V$m$, and draw $m\,n$ perpendicular to it, meeting the axis in $n$.

2. Make V C and V F each equal to R$n$, and F will be the focus of the curve.

3. Take any number of points $r$, $r$, &c. in the axis, through which draw the double ordinates S $r$ S, &c., of an indefinite length.

4. With the radii C F, C$r$, &c., and centre F, describe arcs cutting the corresponding ordinates in the points $s$, $s$, &c., and the curve S V S drawn through all the points of intersection will be the parabola required.

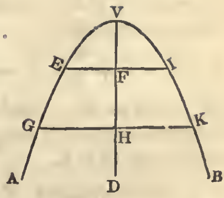
The line $s$ F $s$ passing through the focus F is called the parameter.

### PROBLEM II.

In a parabola, any three of the four following terms being given, viz., any two ordinates and their two abscissas, to find the fourth.

Rule.—As any abscissa is to the square of its ordinate, so is any other abscissa to the square of its ordinate. Or as the square root of any abscissa is to its ordinate, so is the square root of any other abscissa to its ordinate, and conversely.

### EXAMPLES.




The abscissa V F is 18, and its ordinate E F 12; required the ordinate G H, the abscissa of which, V H, is 32.

Here V F = 18, E F = 12, and V H = 32.

Whence $18 : 12^2 :: 32 : \frac{12^2 \times 32}{18} = \frac{144 \times 32}{18} = 256$ = the square of the ordinate G H, and $\sqrt{256} = 16$ = the ordinate required.

2. The abscissa V F is 25 and its ordinate E F 16; required the ordinate G H, the abscissa of which, V H, is 49.

Here $\sqrt{25} : 16 :: \sqrt{49} : \frac{16\sqrt{49}}{\sqrt{25}} = \frac{16 \times 7}{5} = \frac{112}{5} = 22.4$ = the ordinate G H.

3. The abscissa V F is 9, and the ordinates E F 6 and G H 8; required the abscissa V H.

Here $6^2 : 8^2 :: 9 \cdot \frac{8^2 \times 9}{6^2} = \frac{64 \times 9}{36} = 64 \div 4 = 16$ = the abscissa V H.

4. The abscissa V H is 49, and its ordinate G H 22.4; required the ordinate E F, the abscissa of which, V F, is 25.

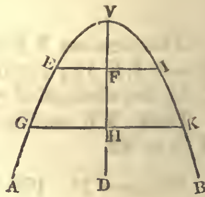
*Ans.* 16, the ordinate E F.

### PROBLEM III.

To find the length of any arc of a parabola cut off by double ordinate.

Rule.—To the square of the ordinate add $\frac{4}{3}$ of the square of the abscissa, and twice the square root of the sum will be the length of the arc nearly.

### EXAMPLES.




1. The abscissa V H is 3, and its ordinate G H 6; what is the length of the arc G V K?

Here $6^2 + \frac{4}{3} \times 3^2 = 36 + \frac{4}{3} \times 9 = 36 + 12 = 48$. And $\sqrt{(48)} \times 2 = 6.9282 \times 2 = 13.8564$ = the length of the arc G V K.

2. The abscissa V H is 6, and its ordinate G H 12; required the length of the arc G V K. *Ans.* 27.7128.

3. The abscissa is 15, and its ordinate 8; what is the length of the arc? *Ans.* 38.1575.

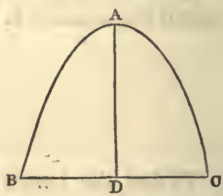
4. The abscissa is 9, and its ordinate 6; what is the length of the arc? *Ans.* 24.

### PROBLEM IV.

To find the area of a parabola, its base and height being given.

Rule.—Multiply the base by the height, and two-thirds of the product will be the area required.

### EXAMPLES.




1. What is the area of the parabola B A C, whose height A D is 12, and the base, or double ordinate, B C, 30?

Here B C = 30, and A D = 12.

Whence $(30 \times 12) \times \frac{2}{3} = 360 \times \frac{2}{3} = 240$ = area required.

2. The abscissa is 18, and the base, or double ordinate, 42; what is the area? *Ans.* 504.

3. What is the area of a parabola whose abscissa is 11, and its double ordinate 21? *Ans.* 154.

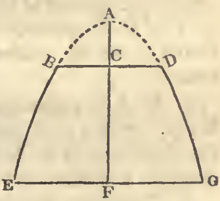
4. What is the area of a parabola whose height is 9, and the base, or double ordinate, 24? *Ans.* 144.

## PROBLEM V.

To find the area of a frustum of a parabola.

RULE.—Divide the difference of the cubes of the two ends of the frustum by the difference of their squares, and this quotient, multiplied by two-thirds of the altitude, will give the area required.

### EXAMPLES.




1. In the parabolic frustum B E G D, the two parallel ends, B D and E G, are 6 and 10, and the altitude, or part of the abscissa, C F, is 3; what is the area?

Here B D $= 6$, E G $= 10$, and C F $= 3$.

Whence $(10^3 - 6^3) \div (10^2 - 6^2) =$ $(1000 - 216) \div (100 - 36) = 784 \div 64 = 12.25$.

Then $12.25 \times (\frac{2}{3}$ of $3) = 12.25 \times 2 = 24.50$, the area.

2. Required the area of a parabolic frustum, the greater end of which is 10, the less 6, and the height 4.2.

*Ans.* 34.3.

3. The greater end of the frustum is 12, the less 10, and the height 2.4; what is the area? *Ans.* $26\frac{26}{55}$.

4. The greater end of the frustum is 24, and the less 20, and the altitude 6; what is the area? *Ans.* 132.3636.

5. Required the area of the parabolic frustum, the greater end of which is 10, the less 6, and the height 5.

*Ans.* $40.8\frac{1}{3}$.

6. The greater end of the frustum is 24, the less end 20, and the height $5\frac{1}{2}$; what is the area? *Ans.* $121\frac{1}{3}$.

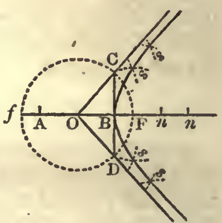
7. Required the area of the parabolic frustum, the greater end of which is 10, the less 6, and the height 4.

*Ans.* $32\frac{2}{3}$.

## THE HYPERBOLA.

### PROBLEM I.

§ 18. To construct an hyperbola, the transverse and conjugate diameters being given.




*Construction.* 1. Make A B the transverse diameter, and C D perpendicular to it, the conjugate.

2. Bisect A B in O, and from O, with the radius O C or O D, describe the circle D *f* C F, cutting A B produced in F and *f*, which points will be the two foci.

3. In A B produced, take any number of points, *n*, *n*, &c., and from F and *f*, as centres, with the distances B *n*, A *n*, as radii, describe arcs cutting each other in *s*, *s*, &c.

4. Through the several points *s*, *s*, &c., draw the curve *s* B *s*, and it will be the hyperbola required.

If straight lines be drawn from the point O, through the extremities C, D of the conjugate diameter C D, they will be the asymptotes of the hyperbola, whose property it is to approach continually to the curve, without ever meeting it.

### PROBLEM II.

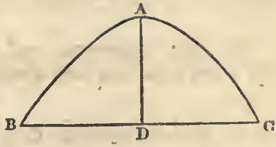
In an hyperbola, any three of the four following terms being given, viz., the transverse and conjugate diameters, an ordinate, and its abscissa, to find the fourth.

Case 1. The transverse and conjugate diameters, and the two abscissas being given, to find the ordinate.

Rule.—As the transverse diameter is to the conjugate, so is the square root of the product of the two abscissas to the ordinate required.

It may be observed that in the hyperbola the less abscissa added to the transverse diameter gives the greater, and the transverse diameter subtracted from the greater abscissa gives the less.

### EXAMPLES.




1. In the hyperbola B A C, the transverse diameter is 50, the conjugate 30, and the less abscissa A D, 12; required the ordinate D C.

Here transverse diameter $=$ 50, conjugate $= 30$, A D $= 12$, and $12 + 50 = 62, =$ the greater abscissa.

Whence $50 : 30 :: \sqrt{(62 \times 12)} : \frac{30}{50}\sqrt{(62 \times 12)} = \frac{3}{5}$ $\sqrt{744} = \frac{3}{5} \times 27.27636 = 16.3658 =$ the ordinate.

2. The transverse diameter is 24, the conjugate 21, and the less abscissa 8; what is the ordinate? *Ans.* 14.

3. The transverse diameter is 36, and the conjugate 24, and the less abscissa 12; what is the ordinate? *Ans.* 16.

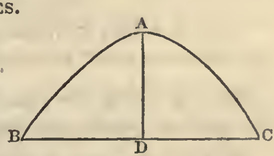
4. The transverse diameter is 120, the conjugate 72, and greater abscissa 160; what is the ordinate? *Ans.* 48.

Case 2.—The transverse and conjugate diameters, and an ordinate being given, to find the two abscissas.

Rule.—As the conjugate diameter is to the transverse, so is the square root of the sum of the squares of the ordinate and semi-conjugate to the distance between the ordinate and centre, or half the sum of the abscissas.

Then the sum of this distance and the semi-transverse will give the greater abscissa, and their difference will give the less.

### EXAMPLES.




1. The transverse diameter is 36, the conjugate 24, and the ordinate, B D, 16; what are the two abscissas?

Here $\sqrt{(16^2 + 12^2)} = \sqrt{(256 + 144)} = \sqrt{400} = 20$.

Then $24 : 36 . : 20 : 30 = \frac{1}{2}$ sum of the abscissas.

And $30 + 18 = 48 =$ the greater abscissa.
Also $30 - 18 = 12 =$ the less abscissa.

2. The transverse diameter is 120, the conjugate 72, and the ordinate 48; what are the two abscissas?

*Ans.* 160 and 40.

3. The transverse and conjugate diameters are 24 and 21; required the two abscissas to the ordinate 14?

*Ans.* 32 and 8.

4. The transverse being 60, and the conjugate 36; required the two abscissas to the ordinate 24.

*Ans.* 80 and 20.

CASE 3.—The transverse diameter, the two abscissas, and the ordinate being given, to find the conjugate.

RULE.—As the square root of the product of the two abscissas is to the ordinate, so is the transverse diameter to the conjugate.

EXAMPLES.

1. The transverse diameter is 36, the ordinate 16, and the two abscissas are 48 and 12; required the conjugate.

Here $\sqrt{(48\times 12)}=\sqrt{576}=24$.

Whence $24 : 16 :: 36 : 24=$ the conjugate required.

2. The transverse diameter is 60, the ordinate 24, and the two abscissas are 80 and 20; required the conjugate.

*Ans.* 36.

3. The transverse diameter is 36, the ordinate 21, and the abscissas 12 and 48; required the conjugate. *Ans.* 31.5.

4. The transverse diameter is 24, the ordinate 14, and the abscissas 8 and 32; required the conjugate. *Ans.* 21.

CASE 4.—The conjugate diameter, the ordinate, and the two abscissas being given, to find the transverse.

RULE 1.—Add the square of the ordinate to the square of the semi-conjugate, and find the square root of their sum.

2. Take the sum or difference of the semi-conjugate and this root, according as the less or greater abscissa is used, and then say: As the square of the ordinate is to the product of the abscissa and conjugate, so is the sum or difference above found to the transverse required.

### EXAMPLES.

1. The conjugate diameter is 21, the ordinate 14, and the less abscissa 8; required the transverse.

Here $\sqrt{(14^2 + 10.5^2)} = \sqrt{(196 + 110.25)} = \sqrt{306.25} = 17.5$, and $17.5 + 10.5 = 28$.

Also $21 \times 8 = 168$.

Whence $196\ (14^2) : 168 :: 28 : 24 =$ the transverse required.

2. The conjugate diameter is 72, the ordinate 48, and the less abscissa 40; what is the transverse? *Ans.* 120.

3. The conjugate diameter is 36, the less abscissa 20, and the ordinate 24; required the transverse. *Ans.* 60.

4. The conjugate diameter is 31.5, the ordinate 21, and the greater abscissa 48; required the transverse.

*Ans.* 36.

### PROBLEM III.

To find the length of any arc of an hyperbola, beginning at the vertex.

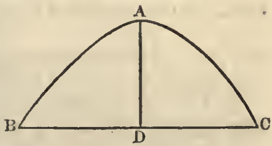
RULE.—To 19 times the square of the transverse add 21 times the square of the conjugate; also to 9 times the square of the transverse add as before 21 times the square of the conjugate; and multiply each of these sums by the abscissa.

2. To each of the two products thus found add 15 times the product of the transverse and the square of the conjugate.

3. Then as the less of these results is to the greater, so is the ordinate to the length of the arc nearly.

### EXAMPLES.

1. In the hyperbola B A C, the transverse diameter is 80, the conjugate 60, the ordinate B D, 10, and the abscissa A D, 2.1637; required the length of the arc B A C.




Here $2.1637 \times [(19 \times 80^2) + (21 \times 60^2)] = 2.1637 \times (121600 + 75600) = 2.1637 \times 197200 = 426681.64$.

And $2.1637 \times [(9 \times 80^2)+(21 \times 60^2)] = 2.1637 \times (57600 + 75600) = 2.1637 \times 133200 = 288204.84$.

Whence $(15 \times 80 \times 60^2) + 426681.64 = 4320000 + 426681.64 = 4746681.64$.

And $(15 \times 80 \times 60^2) + 288204.84 = 4320000 + 288204.84 = 4608204.84$.

Then $4608204.84 : 4746681.64 :: 10 : 10.3005 =$ the length of the arc A C. Therefore, $10.3005 \times 2 = 20.601 =$ the length of the whole arc B A C.

2. The transverse diameter of an hyperbola is 120, the conjugate 72, the ordinate 48, and the abscissa 40; required the whole length of the curve. *Ans.* 125.304.

3. The transverse diameter is 80, the conjugate 60, and the ordinate 16; required the length of the arc.

*Ans.* 17.0856.

4. Required the whole length of the curve of an hyperbola, to the ordinate 20; the transverse and conjugate axis being 80 and 70. *Ans.* 42.267.

## PROBLEM IV.

To find the area of an hyperbola, the transverse, conjugate, and abscissa being given.

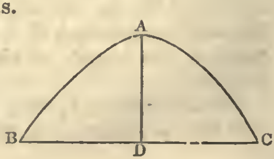
RULE.—1. To the product of the transverse and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.

2. Add 4 times the square root of the product of the transverse and abscissa to the product last found, and divide the sum by 75.

3. Then if 4 times the product of the conjugate and abscissa be divided by the transverse, this last quotient multiplied by the former will give the area required nearly.

### EXAMPLES.

1. In the hyperbola B A C, the transverse axis is 100, the conjugate 60, and the abscissa, or height A D, is 50; required the area.



Here $21\sqrt{[(100\times 50)+(\frac{5}{7}\times 50^2)]}$

$=21\sqrt{(5000+\frac{12500}{7})}=21\sqrt{\frac{47500}{7}}$

$=21\times 82.37544710=1729.8844.$

And $[4\sqrt{(100\times 50)}+1729.8844]\div 75$

$=[(4\times 70.7107)+1729.8844]\div 75$

$=(282.8428+1729.8844)\div 75$

$=2012.7272\div 75=26.836362.$

Whence $[(4\times 50\times 60)\div 100]\times 26.836362=(12000\div 100)\times 26.836362=120\times 26.836362=3220.3634=$ the area required.

2. Required the area of the hyperbola to the abscissa 25, the two axes being 50 and 30. *Ans.* 805.0908.

## PROBLEM V.

To find the area of a space A N O B, bounded on one side by the curve of an hyperbola, by means of equidistant ordinates.


Let A N be divided into a given number of equal parts, A C, C E, &c., and let perpendicular ordinates A B, C D, &c., be erected, and terminated by any hyperbolic curve, B D F, &c.; and let A = A B + N O, B = C D + G H + L M, &c., and C = E F + I K, &c.; then the common distance A C of the ordinates, being multiplied by the sum arising from the addition of A, 4 B, and 2 C, and one-third of the product taken will be the area very nearly.

### EXAMPLES.

1. Given the lengths of 9 equidistant ordinates, 14, 15, 16, 17, 18, 20, 22, 23, and 25 feet, and the common distance 2 feet; what is the area?

Here $[A\,C\times(A+4\,B+2\,C)]\div 3=[2\times(39+300+112)]\div 3=(2\times 451)\div 3=902\div 3=300\frac{2}{3}$ feet.

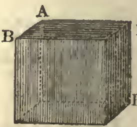
2. Given the lengths of 3 equidistant ordinates, A B = 5 feet, C D = 7, and E F = 8, the length of the base A E = 10; what is the area of the figure A B F E? *Ans.* $68\frac{1}{3}$ ft.

# MENSURATION OF SOLIDS.

DEFINITIONS.

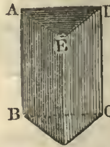
§ **19.** 1. The *measure* of any solid body is the whole capacity or content of that body, when considered under the triple dimensions of length, breadth, and thickness.

2. A *cube* whose side is one inch, one foot, or one yard, &c., is called the *measuring unit;* and the content or solidity of any figure is estimated by the number of cubes of this kind which are contained in it.




3. A *cube* is a solid contained by six equal square sides, or faces, as A B C D E F.


4. A *parallelopipedon* is a solid contained by six rectangular plane faces, every opposite two of which are equal and parallel, as A B C D E F.

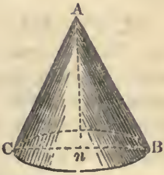



5. A *prism* is a solid whose ends are two equal, parallel, and similar plane figures, and its sides parallelograms; as A B C D E F.

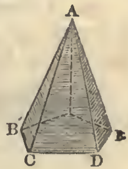
It is called a triangular prism when its ends are triangles; a square prism when its ends are squares; a pentagonal prism, when its ends are pentagons, and so on.


6. A *cylinder* is a solid described by the revolution of a rectangle about one of its sides as an axis, which remains fixed; as A B C D.

7. A *cone* is a solid described by the revolution of a right-angled triangle about one of its legs, which remains fixed; as A B C.

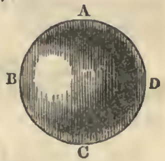



8. A *pyramid* is a solid whose sides are all triangles meeting in a point at the vertex, and the base any plane figure; as A B C D E, which is a pentagonal pyramid.




When the base is a triangle, it is called a *triangular pyramid;* when a square, it is called a *square* or *quadrangular pyramid;* when a pentagon, it is called a *pentagonal pyramid*, &c.

9. A *sphere* is a solid described by the revolution of a semicircle about its diameter, which remains fixed; as A B C D.




10. The centre of a sphere is a point within the figure, equally distant from every point of its convex surface.

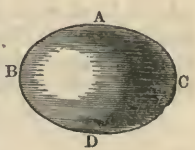
11. A diameter of the sphere is a straight line passing through its centre, and terminated both ways by the convex surface.

And if it be the diameter about which the generating semicircle revolves, it is called the axis of the sphere.

12. A *circular spindle* is a solid generated by the revolution of a segment of a circle about its chord, which remains fixed; as A B D C.


Elliptic, parabolic, and hyperbolic spindles are generated in the same manner as circular spindles, the double ordinate of the section being always considered as fixed.

13. A *spheroid* or *ellipsoid* is a solid generated by the revolution of a semi-ellipsis about one of its axes, which remains fixed; as A B D C.




The spheroid is called *prolate*, when the revolution is made about the transverse axis, and *oblate* when it is made about the conjugate axis.


14. *Parabolic* and *hyperbolic conoids* are solids formed by the revolution of a semi-parabola or semi-hyperbola about its transverse axis, which is considered as quiescent; as A B D; and the same for any other solid of this kind.

15. A *segment* of a pyramid, sphere, or of any other solid, is a part cut off from the top of it by a plane parallel to the base of the figure.

16. A *frustum* or *trunk*, is the part that remains at the bottom, after the segment is cut off.

17. The *zone of a sphere*, is that part which is intercepted between two parallel planes; and when those planes are equally distant from the centre, it is called the middle zone of the sphere.

18. The *height* of a solid is a perpendicular drawn from its vertex to the base, or to the plane on which it is supposed to stand.

19. A *wedge* is a solid, having a rectangular base, and two of its opposite sides meeting in an edge.

20. A *prismoid* is a solid, having on its ends two rect angles parallel to each other, and its upright sides are four trapezoids.

21. An *ungula*, or *hoof*, is a part cut off from a solid by a plane oblique to the base.

The surfaces of all similar solids are to each other as the squares of their like dimensions; such as diameters, circum ferences, like linear sides, &c., &c., and their solidities, as the cubes of those dimensions.

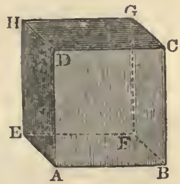
The solidity of cylinders, prisms, parallelopipedons, &c., which have their altitudes equal, are to each other as the squares of their diameters or like sides. The same remark is applicable to frustums of a cone or pyramid when the altitude is the same, and the ends proportional.

## PROBLEM I.

To find the area of the surface of a cube.

Rule.—Multiply the square of the length of one side by the number of sides, and the product will be the area of the surface.

### EXAMPLES.




1. The side of a cube is 18 inches; what is the area of its surface?

Here $(18^2 \times 6) = 324 \times 6 = 1944$ in. $= 13.5$ sq. ft.

2. The side of a cube is 25 inches; what is the area of its surface? *Ans.* $26\frac{1}{24}$ ft.

3. The side of a cube is 12 feet; what is the area of its surface? *Ans.* 864 ft.

4. The side of a cube is 16 feet; what is the area of its surface? *Ans.* 1536 ft.

5. The side of a cube is 19 feet; what is the area of its surface? *Ans.* 2166 ft.

6. The side of a cube is $10\frac{1}{2}$ inches; what is the area of its surface? *Ans.* $4\frac{19}{32}$ ft.

## PROBLEM II.

The area of the surface of a cube being given, to find the length of the side.

Rule.—Divide the area by 6, and extract the square root of the quotient.

### EXAMPLES.

1. The area of a cube is 2400 square inches; what is the length of the side?

Here $\sqrt{(2400 \div 6)} = \sqrt{400} = 20$ inches.

2. The area of a cube is 24 square feet; what is the length of the side? *Ans.* 2 ft.

3. The area of a cube is 216 square feet; what is the length of the side? *Ans.* 6 ft.

4 The area of a cube is 96 feet; what is the length of the side? *Ans.* 4 ft.

5. The area of a cube is 600 square inches; what is the length of the side? *Ans.* 10 in.

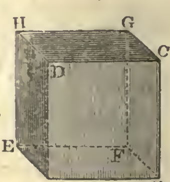
6. The area of a cube is 5400 square inches; what is the length of the side? *Ans.* $2\frac{1}{2}$ ft.

## PROBLEM III.

To find the solidity of a cube, the length of one of its sides being given.

Rule.—Multiply the side by itself, and that product again by the side, or cube the given side, and it will give the solidity required.

### EXAMPLES.




1. The side A B or B C of the cube A B C G H E, is 25.5 inches; what is the solidity?

Here $(A\ B \times B\ C) \times A\ E = (25.5 \times 25.5) \times 25.5 = 650.25 \times 25.5 = 16581.375$ cubic inches.

Or, $(25.5)^3 = 25.5 \times 25.5 \times 25.5 = 16581.375$ cubic inches.

2. What is the solidity of a cube whose side is 5 feet? *Ans.* 125 ft.

3. The side of a cube is 15 inches; what is its solidity? *Ans.* 1.9531 ft.

4. What is the solidity of a cube whose side is 5 feet 3 inches? *Ans.* $144\frac{45}{64}$ ft.

5. How many solid feet are contained in a cubic box whose depth is 32 inches? *Ans.* $18\frac{26}{27}$ ft.

6. How many cubic inches are contained in a cubic piece of timber whose length is 42 inches? *Ans.* 74088 in.

## PROBLEM IV.

To find the side of a cube, the solidity being given.

Rule.—Extract the cube root of the solidity.

### EXAMPLES.

1. What is the length of the side of a cube containing 36 solid feet?

Here $\sqrt[3]{36} = 3.3019$, the side required.

2. What is the length of the side of a cube containing 274 cubic inches? *Ans.* 6.4950 in.

3. What is the length of the side of a cube whose solidity is 1800 cubic inches? *Ans.* 12.1644 in.

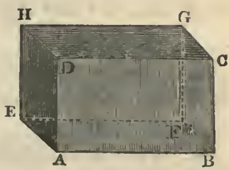
4. What is the length of the side of a cube whose solidity is 789 cubic feet? *Ans.* 9.2404 ft.

## PROBLEM V.

To find the solidity of a parallelopipedon.

Rule.—Multiply the length by the breadth, and that product again by the depth, or altitude, and it will give the solidity required.

### EXAMPLES.




1. Required the solidity of the parallelopipedon A B C G H E, whose length A B is 9 feet, its breadth A E $5\frac{1}{2}$ feet, and its depth, or altitude, A D $7\frac{3}{4}$ feet.

Here $(A\ B \times A\ E \times A\ D) = (9 \times 5.5) \times 7.75 = 49.5 \times 7.75 = 383.625$ solid feet.

2. The length of a parallelopipedon is 8 feet, its breadth 3 feet, and thickness 2 feet; how many solid feet does it contain? *Ans.* 48 ft.

3. The length of a parallelopipedon is 36 inches, the width 20 inches, and depth 18 inches; how many solid feet does it contain? *Ans.* 7.5 feet.

4. The length of a parallelopipedon is 15 feet, and each side of its square base 21 inches; what is the solidity? *Ans.* 45.9375 ft.

5. What is the solidity of a block of marble, whose length is 12 feet, breadth $5\frac{3}{4}$ feet, and depth $2\frac{1}{2}$ feet? *Ans.* $172\frac{1}{2}$ ft.

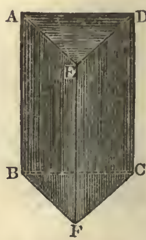
## PROBLEM VI.

To find the solidity of a prism.

Rule.—Multiply the area of the base by the perpendicular height of a prism, and the product will be the solidity.

### EXAMPLES.

1. What is the solidity of the triangular prism A B C D E F, whose length A B is 20 feet, and either of the equal sides B C, C F, or F B, of one of its equilateral ends B C F, 5 feet?




Here, by Problem II., page 82, the area of the base B C F is $= 5^2 \times .433013 = 25 \times .433013 = 10.825325$.

And consequently, $10.825325 \times 20 = 216.5065$ feet, the solidity required.

2. What is the solidity of a triangular prism whose length is 18 feet, and one side of the equilateral end $1\frac{1}{2}$ feet? *Ans.* 17.5370 ft.

3. What is the solidity of a triangular prism whose length is 40 feet, and one side of the equilateral end 18 inches? *Ans.* 38.9711 ft.

4. What is the solidity of a triangular prism whose length is 24 feet, and one side of the equilateral end 16 inches? *Ans.* 18.4752 ft.

5. What is the value of a prism whose height is 32 feet, and each side of the equilateral end 14 inches, at 20 cents per solid foot? *Ans.* \$3.772.

6. Required the solidity of a prism whose base is a hexagon, supposing each of the equal sides to be 1 foot 6 inches, and the length of the prism 16 feet. *Ans.* 93.5307 ft.

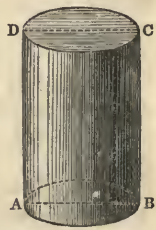
### PROBLEM VII.

To find the convex surface of a cylinder.

RULE.—Multiply the circumference or periphery of the base, by the height of the cylinder, and the product will be the convex surface required; to which add the area of each end, and the sum will be the whole surface of the cylinder

#### EXAMPLES.

1. What is the convex surface of the right cylinder A B C D, whose length, B C, is 24 feet, and the diameter of its base, A B, 16 feet?




Here $3.1416 \times 16 = 50.2656$, the circumference of the base.

And $50.2656 \times 24 = 1206.3744$ square feet, the convex surface required.

2. What is the whole surface of a right cylinder, the diameter of whose base is $2\frac{1}{2}$ feet, and the height 5 feet?

Here $3.1416 \times 2.5 = 7.854$, the circumference of the base.

And $7.854 \times 5 = 39.27$ square feet, the convex surface.

Then $(2.5^2 \times .7854) \times 2 = (6.25 \times .7854) \times 2 = 4.90875 \times 2 = 9.8175$ square feet, the area of the ends.

Whence $39.27 + 9.8175 = 49.0875$ square feet, the whole surface.

3. Required the convex surface of a right cylinder, whose circumference is 8 feet 4 inches, and its length 18 feet.
*Ans.* 150 ft.

4. What is the convex surface of a right cylinder, the diameter of whose base is 2 feet, and its length 30 feet?
*Ans.* 188.496 ft.

5. What is the whole surface of a right cylinder, the diameter of whose base is 16 inches and its height 20 feet?

*Ans.* 86.5685 ft.

6. How many square yards are contained in the whole surface of a cylinder, the diameter of whose base is 4 feet and its length 10 feet?

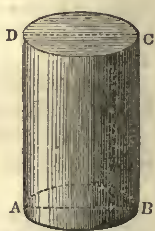
*Ans.* 16.7552 yds.

## PROBLEM VIII.

To find the solidity of a cylinder.

RULE.—Multiply the area of the base by the perpendicular height of the cylinder, and the product will be the solidity.

### EXAMPLES.




1. What is the solidity of the cylinder A B C D, the diameter of whose base, A B, is 30 inches, and the height, B C, 55 inches?

Here $(30^2 \times .7854) \times 55 = (900 \times .7854) \times 55 = 706.86 \times 55 = 38877.3$ cubic inches.

2. What is the solidity of a cylinder, whose height is 30 feet, and the circumference of its base 20 feet?

Here $(20^2 \times .07958) \times 30 = (400 \times .07958) \times 30 = 31.832 \times 30 = 954.96$ cubic feet.

3. What is the solidity of a cylinder whose height is 4 feet, and the diameter of its base 10 inches?

*Ans.* 2.1816 ft.

4 The diameter of a cylinder is 16 inches, and the length 20 feet; what is the solidity?

*Ans.* 27.9253 ft.

5. The circumference of a cylinder is 2 feet, and the length 5 feet; what is the solidity?

*Ans.* 1.5916 ft.

6. The circumference of a cylinder is 20 feet, and the height 19.318 feet; what is the solidity?

*Ans.* 614.9305 ft.

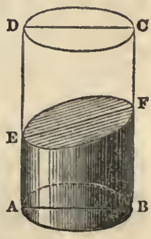
## PROBLEM IX.

To find the curve surface of a cylindric ungula, when the section passes obliquely through the sides of the cylinder.

Rule.—Multiply the circumference of the base by half the sum of the greatest and the least heights of the ungula, and the product will be the curve surface.

### EXAMPLES.

1. What is the curve surface of a cylindric ungula, the diameter of whose base is A B, 16 feet, and the greatest and least heights are B F, 17, and A E, 14 feet?




Here $3.1416 \times 16 = 50.2656$, the circumference of the base.

Then $50.2656 \times (17 + 14) \div 2 = 50.2656 \times (31 \div 2) = 50.2656 \times 15.5 = 779.1168$ feet, the curve surface required.

2. What is the curve surface of a cylindric ungula, the circumference of whose base is 21 feet, and the greatest and least heights are 13 and 8 feet? *Ans.* 220.5 ft.

3. What is the curve surface of a cylindric ungula, the diameter of whose base is 19 feet, and the greatest and least heights are $13\frac{3}{4}$ and $11\frac{1}{4}$ feet? *Ans.* 746.13 ft.

## PROBLEM X.

To find the solidity of a cylindric ungula, when the section passes obliquely through the opposite sides of the cylinder.

Rule.—Multiply the area of the base of the cylinder by half the sum of the greatest and least heights of the ungula, and the product will be the solidity.

### EXAMPLES.

1. What is the solidity of a cylindric ungula, the diameter of whose base, A B, is 12 feet, and the greatest and least heights are B F, 6, and A E, 4 feet?

Here $(12^2 \times .7854) = 144 \times .7854 = 113.0976$ square feet, the area of the base.

Then $113.0976 \times (6+4) \div 2 = 113.0976 \times (10 \div 2) = 113.0976 \times 5 = 565.488$ feet, the solidity required.

2. What is the solidity of a cylindric ungula, the diameter of the base of which is 10 feet, and the greatest and least heights are 4 and 3 feet? *Ans.* 274.89 ft.

3. What is the solidity of a cylindric ungula, the circumference of the base of which is 24 feet, and the greatest and least heights 18 and 12 feet? *Ans.* 687.5712 feet.

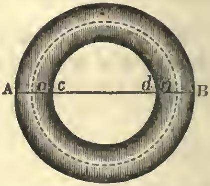
## PROBLEM XI.

To find the convex superficies of a cylindric ring.

Rule.—To the thickness of the ring add the inner diameter, and this sum being multiplied by the thickness, and the product again by 9.8696 (or the square of 3.1416) will give the superficies required.

### EXAMPLES.

1. The thickness, A $c$, of a cylindric ring is 3 inches, and the inner diameter, $cd$, 12 inches; what is the convex superficies?




Here $[(12+3) \times 3] \times 9.8696 = (15 \times 3) \times 9.8696 = 45 \times 9.8696 = 444.132$ square inches.

2. The thickness of a cylindric ring is 4 inches, and the inner diameter 18 inches; what is the convex superficies? *Ans.* 868.5248 inches.

3. The thickness of a cylindric ring is 2 inches, and the inner diameter 1 foot 6 inches; what is the convex superficies? *Ans.* 394.784 inches

4. The thickness of a cylindric ring is 3 inches, and its inner diameter 9 inches; what is the convex superficies?
*Ans.* 355.3056 inches.

5. The thickness of a cylindric ring is 2 inches, and the inner diameter 12 inches; what is the convex superficies?
*Ans.* 276.3488 inches.

6. The thickness of a cylindric ring is 3.5 inches, and its inner diameter 18.765 feet; what is its convex superficies?
*Ans.* 7899.4304 inches.

## PROBLEM XII.

To find the solidity of a cylindric ring.

RULE.—To the thickness of the ring add the inner diameter, and this sum being multiplied by the square of half the thickness, and the product again by 9.8696, will give the solidity.

### EXAMPLES.

1. What is the solidity of an anchor ring, whose inner diameter is 8 inches, and thickness in metal 3 inches?
Here $[(8+3) \times 1.5^2] \times 9.8696 = (11 \times 2.25) \times 9.8696 = 24.75 \times 9.8696 = 244.2726$ cubic inches.

2. What is the solidity of an anchor ring whose inner diameter is 9 inches, and the thickness of metal 3 inches?
*Ans.* 266.4792 inches.

3. The inner diameter of a cylindric ring is 12 inches, and its thickness 4 inches; what is its solidity?
*Ans.* 631.6544 inches.

4. Required the solidity of a cylindric ring whose thickness is 2 inches, and its inner diameter 16 inches.
*Ans.* 177.6528 inches.

5. Required the solidity of a cylindric ring whose inner diameter is 12 inches, and thickness 5 inches.
*Ans.* 1048.645 inches.

6. What is the solidity of a cylindric ring whose thickness is 4 inches, and inner diameter 16 inches?
*Ans.* 789.568 inches.

### PROBLEM XIII.

The solidity and thickness of a cylindric ring being given, to find the inner diameter.

RULE.—Divide the solidity by 9.8696, and that quotient by the square of half the thickness, from which subtract the thickness, and the remainder will be the inner diameter of the ring.

#### EXAMPLES.

1. The thickness of a cylindric ring is 4 inches, and its solidity 789.568 solid inches. What is its inner diameter?

Here $789.568 \div 9.8696 = 80$, and $80 \div 2^2 = 80 \div 4 = 20$, then $20 - 4 = 16$ inches the diameter.

2. Required the inner diameter of a cylindric ring whose solidity is 138.1744 inches, and thickness 2 inches.

*Ans.* 12 inches.

3. What is the inner diameter of a cylindric ring whose solidity is 1 solid foot, and thickness 4 inches?

*Ans.* 39.77 inches.

4. If the solidity of a cylindric ring be 4 solid feet, and the thickness 3.5 inches, what is the inner diameter?

*Ans.* 18.765 feet.

5. What must be the inner diameter of a cylindric ring whose solidity is 1 solid inch, and thickness $\frac{1}{8}$ of an inch?

*Ans.* 25.8132 inches.

6. What is the inner diameter of a cylindric ring whose solidity is 244.2726 inches, and the thickness 3 inches?

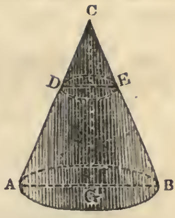
*Ans.* 8 inches.

### PROBLEM XIV.

To find the surface of a right cone or pyramid.

RULE.—Multiply the circumference or perimeter of the base by the slant height or length of the side of the cone or pyramid, and half the product will be the surface required. And if this be added to the area of the base, it will give the whole surface.

EXAMPLES.




1. The diameter of the base A B, of a right cone C A B, is 6 feet, and the slant height A C or B C, 21 feet. Required the convex surface of the cone.

Here $3.1416 \times 6 = 18.8496 =$ the circumference of the base.

And $(18.8496 \times 21) \div 2 = 395.8416 \div 2 = 197.9208$ square feet, the convex surface required.

2. The circumference of a right cone is 10 feet, and the slant height 12 feet. What is the whole surface of the cone?

Here $(10 \times 12) \div 2 = 120 \div 2 = 60$ square feet, the convex surface.

And $(10^2 \times .07958) = 100 \times .07958 = 7.958$ square feet, the area of base.

Whence $60 + 7.958 = 67.958$ square feet, the whole surface required.

3. Required the whole surface of a triangular pyramid, each side of its base being $5\frac{1}{2}$ feet, and its slant height $17\frac{1}{2}$ feet.

Here $5.5 \times 3 = 16.5 =$ the perimeter of the base.

And $(16.5 \times 17.5) \div 2 = 288.75 \div 2 = 144.375$ square feet, the outward surface of the pyramid.

Also $(5.5^2 \times .433013) = 30.25 \times .433013 = 13.0986$ square feet, the area of the base.

Whence $144.375 + 13.0986 = 157.4736$ square feet, the whole surface required.

4. The slant height of a right cone is 20 feet, and the diameter of the base 8 feet; required the convex surface.

*Ans.* 251.328 ft.

5. The circumference of a right cone is 27.5 feet, and the slant height 11 feet; required the convex surface.

*Ans.* 151.25 ft.

6. The slant height of a right cone is 20 feet, and the diameter 3 feet; what is the whole surface?

*Ans.* 101 3166 ft

7. Required the outward surface of a triangular pyramid, each side of its base being $3\frac{1}{2}$ feet, and its slant height 14 feet. *Ans.* 73.5 ft.

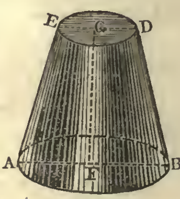
8. The circumference of a right cone is 10 feet, and the perpendicular height 12 feet; required the convex surface. *Ans.* 60.525 ft.

### PROBLEM XV.

To find the surface of the frustum of a right cone or pyramid.

RULE.—Multiply the sum of the perimeters of the two ends by the slant height of the frustum, and half the product will be the surface required.

EXAMPLES.




1. In the frustum of the cone A B D E, the circumferences of the two ends A B and E D, are 22.75 and 15.5 feet respectively, and the slant height, A E, is 26 feet; required the convex surface.

Here $[(22.75 + 15.5) \times 26] \div 2 = (38.25 \times 26) \div 2 = 994.5 \div 2 = 497.25$ square feet, the convex surface required.

2. Required the surface of the frustum of a square pyramid, one side of the base being $12\frac{1}{2}$ feet, and of the upper end $5\frac{3}{4}$ feet, and its slant height $40\frac{1}{4}$ feet.

Here $12\frac{1}{2} \times 4 = 50 =$ the perimeter of the base, and $5\frac{3}{4} \times 4 = 23 =$ the perimeter of the upper end.

Then $[(50 + 23) \times 40.25] \div 2 = (73 \times 40.25) \div 2 = 2938.25 \div 2 = 1469.125$ square feet, the surface required.

3. What is the convex surface of the frustum of a right cone, the circumference of the greater end being $23\frac{3}{4}$ feet, and that of the less end $16\frac{1}{4}$ feet, and the length of the slant side 12 feet? *Ans.* 240 ft.

4. What is the convex surface of the frustum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the slant side 20 feet? *Ans.* 376.992 ft.

5. Required the surface of a hexagonal pyramid, one side of the base being $8\frac{1}{3}$ feet, and of the upper end $3\frac{2}{3}$ feet, and the slant height $20\frac{3}{8}$ feet. *Ans.* 733.5 ft.

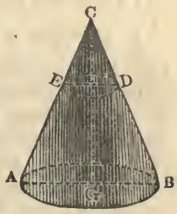
6. What is the convex surface of the frustum of a right cone, the diameters of the ends being 5 and 4 feet, and the slant height 6 feet? *Ans.* 84.8232 ft.

## PROBLEM XVI.

To find the solidity of a cone or pyramid.

RULE.—Multiply the area of the base by the perpendicular height of the cone or pyramid, and one-third of the product will be the solidity.

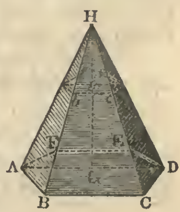
### EXAMPLES.




1. Required the solidity of a cone C A B, whose diameter, A B, is 30 feet, and its perpendicular height, G C, 36 feet.

Here $(.7854 \times 30^2) = .7854 \times 900 = 706.86 =$ the area of the base.

And $(706.86 \times 36) \div 3 = 25446.96 \div 3 = 8482.32$ feet, the solidity.




2. Required the solidity of the hexagonal pyramid H A D H, each of the equal sides of its base being 20 feet, and the perpendicular height, H G, 50 feet.

Here, by the table, page 82, for polygons we have 2.598076 (multiplier when the side is 1) $\times 20^2 = 2.598076 \times 400 = 1039.2304 =$ the area of the base.

And $(1039.2304 \times 50) \div 3 = 51961.52 \div 3 = 17320.5066$ feet, the solidity.

3. What is the solidity of a cone, the diameter of whose base is 18 inches, and its altitude 24 feet? *Ans.* 14.1372 ft.

4. If the circumference of the base of a cone be 60 feet, and its height 72 feet; what is the solidity?

*Ans.* 6875.712 ft.

5. What is the solidity of a triangular pyramid, whose height is 30 feet, and each side of the base 3 ft.?

*Ans.* 38.9711 ft.

6. What is the solidity of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?

*Ans.* 27.5276 ft.

7. What is the solidity of a cone, whose diameter of the base is 14 feet, and the slant side being 25 feet?

*Ans.* 1231.5072 ft.

## PROBLEM XVII.

To find the solidity of the frustum of a cone or pyramid.

1. For the frustum of a cone, the diameters of the two ends and the height being given.

Rule.—Divide the difference of the cubes of the diameters of the two ends by the difference of the diameters, and this quotient being multiplied by .7854, and again by one-third of the height, will give the solidity.

Or divide the difference of the cubes of the circumferences of the two ends, by the difference of the circumferences, and the quotient being multiplied by .07958, and again by one-third of the height, will give the solidity.

Or to the product of the diameters add one-third of the square of their difference, and that sum being multiplied by .7854, and again by the height, will give the solidity.

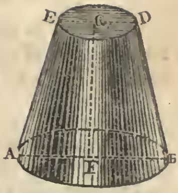
Or to the product of the circumferences add one-third of the square of their difference, and this sum being multiplied by .07958, and again by the height, will give the solidity.

2. For the frustum of a pyramid, the sides of the base and the height being given.

Rule.—To the areas of the two ends of the frustum add the square root of their product, and this sum being multiplied by one-third of the height, will give the solidity.

EXAMPLES.

1. What is the solidity of the frustum of the cone A B D E, the diameter of whose greater end, A B, is 6 feet, that of the less end, E D, 4 feet, and the perpendicular height, F G, 9 feet?




Here $(6^3 - 4^3) \div (6 - 4) = (216 - 64) \div 2 = 152 \div 2 = 76$.

And $(76 \times .7854) \times (9 \div 3) = 59.6904 \times 3 = 179.0712$ feet, the solidity required.

Or $[(6 \times 4) + (6 - 4)^2 \div 3] = 24 + (2^2 \div 3) = (24 + \frac{4}{3}) = 25\frac{1}{3}$. And $(25\frac{1}{3} \times .7854) \times 9 = 19.8968 \times 9 = 179.0712$ feet, the solidity, as before.

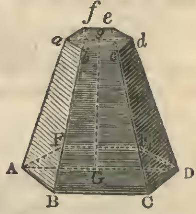
2. What is the solidity of the frustum of a cone, the circumference of the greater end being 40 feet, and that of the less 20 feet, and the length or height 51 feet?

Here $(40^3 - 20^3) \div (40 - 20) = (64000 - 8000) \div 20 = 56000 \div 20 = 2800$.

And $(2800 \times .07958) \times (51 \div 3) = 222.824 \times 17 = 3788.008$ feet, the solidity required.

Or $[(40 \times 20) + (40 - 20)^2 \div 3] = 800 + (20^2 \div 3) = 800 + 133\frac{1}{3} = 933\frac{1}{3}$, and $(933\frac{1}{3} \times .07958) \times 51 = 74.274666 \times 51 = 3788.008$ feet, the solidity, as before.

3. What is the solidity of the frustum $aADd$ of an hexagonal pyramid, the side A B of whose greater end is 4 feet, that $ab$ of the less end 3, and the height G $g$ 9 feet?




Here 2.598076 (the tabular multiplier) $\times 3^2 = 2.598076 \times 9 = 23.382684$ the area of the less end.

And $2.598076 \times 4^2 = 2.598076 \times 16 = 41.569216$ the area of the greater end.

Whence $\sqrt{(23.382684 \times 41.569216)} = \sqrt{971.999841} = 31.176912$.

And $(23.382684 + 41.569216 + 31.176912) \times (9 \div 3) = 96.128812 \times 3 = 288.386436$ feet.

4. What is the solidity of the frustum of a cone, the diameter of the greater end being 4 feet, that of the less end 2, and the altitude 9 feet? *Ans.* 65.9736 feet

5. What is the solidity of the frustum of a cone, the diameter of the greater end of which is 5 feet, that of the less end 3 feet, and the altitude 4 feet? *Ans.* 51.3128 feet.

6. What is the solidity of the frustum of a cone, the circumference of the greater end being 20 feet, that of the less end 10 feet, and the length or height 21 feet?

*Ans.* 389.942 feet.

7. What is the solidity of the frustum of a cone, the circumference of the greater end being 12 feet, that of the less end 8 feet, and the height 5 feet? *Ans.* 40.3205 feet.

8. What is the solidity of the frustum of a square pyramid, one side of the greater end being 18 inches, that of the less end 15 inches, and the altitude 60 inches?

*Ans.* 9.4791 feet.

9. What is the solidity of the frustum of an equilateral triangular pyramid, one side of the greater end being 14 inches, that of the less end 8 inches, and the height 10 feet?

*Ans.* 3.7287 feet.

10. What is the solidity of the frustum of a pentagonal pyramid, the side of whose greater end is 18 inches, that of the less end 12 inches, and the height 4 feet 6 inches?

*Ans.* 12.2583 feet

### PROBLEM XVIII.

The solidity and altitude of a cone being given, to find the diameter.

Rule.—Divide the solidity by the product of .7854, and one-third of the altitude, and the square root of the quotient will be the diameter.

#### EXAMPLES.

1. The solidity of a cone is 16 feet, and the altitude 9 feet; what is the diameter?

Here $\sqrt{\{16 \div [.7854 \times (9 \div 3)]\}} = \sqrt{[16 \div (.7854 \times 3)]}$ $= \sqrt{(16 \div 2.3562)} = \sqrt{6.7906} = 2.6057$ feet, the diameter.

2. The altitude of a cone is 15 feet, and the solidity 30 feet; what is the diameter? *Ans.* 2.7639 feet.

3. The solidity of a cone is 18 feet, and the altitude 8 feet; what is the diameter? *Ans.* 2.9316 feet.

## PROBLEM XIX.

The solidity and diameter of a cone being given, to find the altitude.

RULE.—Divide the solidity by the product of .7854 and the square of the diameter, and the quotient, being multiplied by 3, will give the altitude.

### EXAMPLES.

1. The solidity of a cone is 30 feet, and the diameter 2 feet; what is the altitude?

Here $[30 \div (.7854 \times 2^2)] \times 3 = [30 \div (.7854 \times 4)] \times 3 = (30 \div 3.1416) \times 3 = 9.5492 \times 3 = 28.6476$ feet, the altitude.

2. The diameter of a cone is 20 inches; what must be the altitude, to make 20 solid feet? *Ans.* 27.5019 ft.

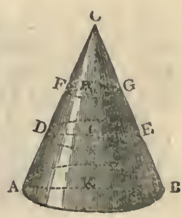
3. The solidity of a cone is 2513.28 feet, and the diameter 20 feet; what is the altitude? *Ans.* 24 ft.

## PROBLEM XX.

The altitude of a cone or pyramid being given, to divide it into two or more equal parts, by sections parallel to the base, to find the perpendicular height of each part.

RULE.—Multiply the cube of the altitude by the numerator of the proportion left at the vertex, and divide the product by the denominator; the cube root of the quotient will be the altitude of the cone or pyramid left at the vertex.

### EXAMPLES.




1. The altitude of the cone A B C is 10 feet, to be divided into three equal parts by sections parallel to the base; required the perpendicular height of each part.

Here $\sqrt[3]{[(10^3 \times 1) \div 3]} = \sqrt[3]{[(1000 \times 1) \div 3]} = \sqrt[3]{(1000 \div 3)} = \sqrt[3]{333\frac{1}{3}} = 6.9336$ feet, the altitude of the first section $=$ C H.

And $\sqrt[3]{[(10^3 \times 2) \div 3]} = \sqrt[3]{[(1000 \times 2) \div 3]} = \sqrt[3]{(2000 \div 3)} = \sqrt[3]{666\frac{2}{3}} = 8.7358$ feet, the altitude C I. Now C I — C H $= 8.7358 - 6.9336 = 1.8022$ feet, the altitude of the second section I H.

Then C K — C I $= 10 - 8.7358 = 1.2642$ feet, the altitude of the third section I K.

2. The altitude of a pyramid is 12 feet, to be divided into three equal parts by sections parallel to the base; required the perpendicular height of each part.

*Ans.* 8.3203, 2.1626, and
1.5171 feet, the altitudes.

3. The altitude of a cone is 20 feet, to be divided into four equal parts by sections parallel to the base; required the perpendicular height of each part.

*Ans.* 12.5992, 3.2748, 2.2972, and
1.8288 feet, the altitudes.

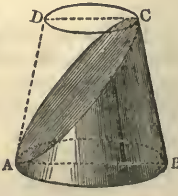
## PROBLEM XXI.

To find the solidity of an ungula when the section passes through the opposite extremities of the ends of the frustum.

RULE.—From the square of the greater diameter subtract the square root of the product of the two diameters, multiplied by the less diameter.

This difference being divided by the difference of the diameters, and the quotient, multiplied by the greater diameter, that product by the height, and the last product by .2618 will give the solidity.

### EXAMPLES.




1. Required the solidity of a conical ungula, the diameter of the greater end being 5 feet, that of the less end 1.8 feet, and the height 12 feet.

Here $[5^2 - 1.8\sqrt{(5 \times 1.8)}] \div (5 - 1.8) = [(25 - 1.8\sqrt{9}) \div 3.2] = [25 - (1.8 \times 3) \div 3.2] = [(25 - 5.4) \div 3.2] = (19\ 6 \div 3.2) = 6.125$.

And $(6.125 \times 5 \times 12 \times .2618) = 96.2115$ feet, the solidity.

2. Required the solidity of a conical ungula, the diameter of the greater end being 10 feet, that of the less end $2\frac{1}{2}$ feet, and the height 15 feet. *Ans.* 458.15 ft.

3. Required the solidity of a conical ungula the diameter of the greater end being 4.23 inches, that of the less end 3.7 inches, and the height 5.7 inches.

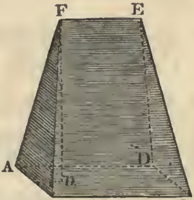
*Ans.* 38.7692 in.

### PROBLEM XXII.

To find the solidity of a cuneus or wedge.

Rule.—Add twice the length of the base to the length of the edge, then multiply this sum by the height of the wedge, and again by the breadth of the base, and one sixth of the last product will be the solidity.

#### EXAMPLES.




1. How many solid feet are there in a wedge whose base is 5 feet 4 inches long, and 9 inches broad, the length of the edge being 3 feet 6 inches, and the perpendicular height 2 feet 4 inches?

Here 5 ft. 4 in. = 64 in., 3 feet 6 in. = 42 in. and 2 ft. 4 in. = 28 in.

Then $(64 \times 2 + 42) \times 28 = (128 + 42) \times 28 = 170 \times 28 = 4760$.

And $(4760 \times 9) \div 6 = 42840 \div 6 = 7140$ solid inches.

Whence $7140 \div 1728 = 4.1319$ solid feet.

2. The length and breadth of the base of a wedge are 35 and 15 inches, the length of the edge 55 inches, and the perpendicular height 18 inches; what is the solidity? *Ans.* 3.2552 ft.

3. The length and breadth of the base of a wedge are 27 and 8 inches, the length of the edge 36 inches, and the perpendicular height 3 feet 6 inches; what is the solidity? *Ans.* 2.9166 ft.

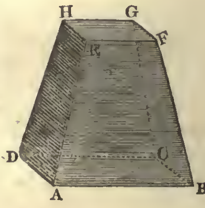
## PROBLEM XXIII.

To find the solidity of a prismoid.

Rule.—Multiply the sum of the lengths of the two ends by the sum of the breadths, and add this product to the sum of the areas of the two ends; multiply the result by one sixth of the height, and the product will be the solidity.

### EXAMPLES.

1. What is the solidity of a rectangular prismoid, the length and breadth of one end being 14 and 12 inches, and the corresponding sides of the other 6 and 4 inches; and the perpendicular $30\frac{1}{2}$ feet?




Here (by Prob. I, p. 61) $EF \times EH = 6 \times 4 = 24$ square inches, the area of the less end, E F G H.

And, by the same problem $AB \times AD = 14 \times 12 = 168$ square inches, the area of the greater end, A B C D.

Also $30\frac{1}{2}$ft. $= 366$ inches. calling the height $h$,

$$[(AB + EF) \times (AD + EH) + (EFGH + ABCD)] \times \frac{h}{6} = [(14 + 6) \times (12 + 4) + (168 + 24)] \frac{366}{6}.$$
$$= [(20 \times 16) + 192] \times 61 = (320 + 192) \times 61 = 512 \times 61$$
$= 31232$ cubic inches, the solidity required.

2. What is the solidity of a rectangular prismoid, the length and breadth of one end being 12 and 8 inches, and the corresponding sides of the other 8 and 6 inches, and the perpendicular height 5 feet? *Ans.* 2.4537 ft.

3. What is the solidity of a stick of hewn timber, whose ends are respectively 30 by 27 inches, and 24 by 18 inches, and whose length is 48 feet? *Ans.* 204 ft.

4. What is the capacity of a coal wagon, whose inside dimensions are as follows: at the top the length is 7 feet, and breadth 6 feet, at the bottom the length is 5 feet, and breadth 3 feet, and the perpendicular depth is 4 feet? *Ans.* 110 ft

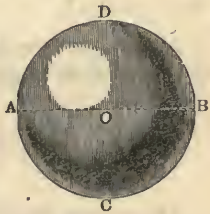
## PROBLEM XXIV.

To find the convex surface of a sphere.

Rule.—Multiply the diameter of the sphere by its circumference, and the product will be the convex superficies required.

The curve surface of any zone or segment will also be found by multiplying its height by the whole circumference of the sphere.

### EXAMPLES.




1. What is the convex surface of a globe A D B C, whose diameter, A B, is 16 inches?

Here $(3.1416 \times 16) \times 16 = 50.2656 \times 16 = 804.2496$ square inches, the surface required.

2. What is the convex surface of a sphere whose diameter is 10 feet? *Ans.* 314.16 ft.

3. What is the convex surface of a sphere whose diameter is 4 feet? *Ans.* 50.2656 ft.

4. The diameter of a globe is 21 inches; what is the convex surface of that segment of it whose height is $4\frac{1}{2}$ inches? *Ans.* 296.8812 inches.

5. What is the convex surface of a sphere whose diameter is 6 feet? *Ans.* 113.0976 ft.

6. If the diameter of the globe we inhabit be 7935 miles; what is the convex surface? *Ans.* 197808409.26 miles.

## PROBLEM XXV.

To find the solidity of a sphere or globe.

Rule.—Multiply the cube of the diameter by .5236, and the product will be the solidity.

### EXAMPLES.

1. What is the solidity of a globe whose diameter is 4.5 feet?

Here $(4.5^3 \times .5236) = 4.5 \times 4.5 \times 4.5 \times .5236 = 91.125 \times .5236 = 47.71305$ solid feet.

2. What is the solidity of a globe whose diameter is $3\frac{1}{2}$ feet? *Ans.* 22.4493 ft.

3. What is the solidity of a globe whose diameter is 17 inches? *Ans.* 1.4886 ft.

4. What is the solidity of a globe whose diameter is 3 feet 4 inches? *Ans.* 19.3925 ft.

5. How many cubic miles are contained in the solidity of the earth, if its diameter be 7935 miles?

*Ans.* 261601621246.35 miles.

## PROBLEM XXVI.

The convex surface of a globe being given, to find its diameter.

RULE.—Multiply the given area by .31831, and the square root of the product will be the diameter.

### EXAMPLES.

1. What is the diameter of that globe, the area of whose convex surface is 14 square feet?

Here $\sqrt{(14 \times .31831)} = \sqrt{4.45634} = 2.1110$ feet, the diameter required.

2. The convex surface of a sphere is one square rood; what is its diameter? *Ans.* 3.5682 rods.

3. The expense of gilding a ball at \$1.80 per square foot is thirty-four dollars; what is its diameter?

*Ans.* 2.452 ft.

## PROBLEM XXVII.

The solidity of a globe being given, to find the diameter.

RULE.—Divide the solidity by .5236, and extract the cube root of the quotient.

### EXAMPLES.

1. The solidity of a globe is 2000 solid inches; what is its diameter?

Here $\sqrt[3]{(2000 \div .5236)} = \sqrt[3]{3819.7097} = 15.631$ inches the diameter.

2. The solidity of a globe is 10 solid feet; what is its diameter? *Ans.* 2.67 ft.

3. What is the circumference of a globe whose solidity is 8 solid feet? *Ans.* 7.7952 ft.

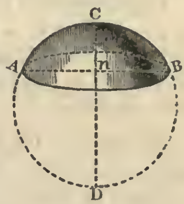
## PROBLEM XXVIII.

To find the solidity of the segment of a sphere.

RULE.—To three times the square of the radius of its base add the square of its height; and this sum multiplied by the height, and the product again by .5236, will give the solidity.

Or, from three times the diameter of the sphere subtract twice the height of the segment, multiply by the square of the height, and that product by .5236; the last product will be the solidity.

### EXAMPLES.




1. The radius A$n$ of the base of the segment CAB is 7 inches, and the height C$n$ 4 inches; what is the solidity?

Here $[(7^2 \times 3) + 4^2] \times 4 = [(49 \times 3) + 16] \times 4 = (147 + 16) \times 4 = 163 \times 4 = 652$.

Then $652 \times .5236 = 341.3872$ solid inches

2. The diameter of a sphere is 6 inches. What is the solidity of the segment whose height is 2 inches?

Here $[(6 \times 3) - (2 \times 2)] \times 2^2 = (18 - 4) \times 4 = 14 \times 4 = 56$. Then $56 \times .5236 = 29.3216$ solid inches.

3. What is the solidity of a spherical segment, the diameter of its base being 40 inches, and the height 10 inches?

*Ans.* 3.9391 feet.

4. The diameter of a sphere is 18 inches; what is the solidity of a segment cut from it, the height being 3 inches?

*Ans.* 226.1952 inches.

5. The diameter of a spherical segment is 20 inches, and the height 6 inches; how many gallons of water will it hold, each gallon containing 282 cubic inches?

*Ans.* 3.743 gallons.

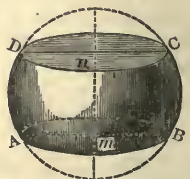
## PROBLEM XXIX.

To find the solidity of a frustum or zone of a sphere.

Rule.—To the sum of the squares of the radii of the two ends, add one-third of the square of their distance, or of the breadth of the zone, and this sum multiplied by the said breadth, and the product again by 1.5708, will give the solidity.

### EXAMPLES.

1. What is the solidity of the zone ABCD, whose greater diameter, AB, is 1 foot 8 inches, the less diameter, DC, 1 foot 3 inches, and the distance $nm$ of the two ends 10 inches?




Here $[(Am^2 + Dn^2) + \frac{1}{3}(nm)^2] \times nm \times 1.5708 = [(10^2 + 7\frac{1}{2}^2) + (10^2 \div 3)] \times 10 \times 1.5708 = (100 + 56\frac{1}{4} + 33\frac{1}{3}) \times 15.708 = 189\frac{7}{12} \times 15.708 = 2977.975$ cubic inches the solidity of the zone required.

2. What is the solidity of a zone whose greater diameter is 9 feet 3 inches, less diameter 6 feet 9 inches, and height 5 feet 6 inches. *Ans.* 370.3242 feet.

3 What is the solidity of a zone, whose greater diameter

is 2 feet, the less diameter 1 foot 8 inches, and the distance of the ends 4 inches? *Ans.* 1566.6112 inches.

4. Required the solidity of the middle zone of a sphere, whose top and bottom diameters are each 3 feet, and the breadth of the zone 4 feet? *Ans.* 61.7848 feet.

## PROBLEM XXX.

To find the solidity of a circular spindle, its length and middle diameter being given.

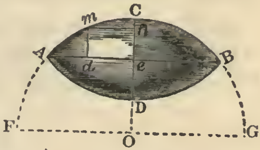
RULE.—To the square of half the length of the spindle, or longest diameter, add the square of half the middle diameter, and this sum divided by the middle diameter will give the radius of the circle.

2. Take half the middle diameter from the radius thus found, and it will give the central distance; or that part of the radius that lies between the centre of the circle, and that of the spindle.

3. Find the area of the generating circular segments, by Problem XI., rule 5, page 104.

4. From one-third of the cube of half the length of the spindle, subtract the product of the central distance, and half the area last mentioned, and the remainder multiplied by 12.5664, will give the solidity of the spindle.

### EXAMPLES.




1. The longest diameter, AB, of the circular spindle ADBC, is 48 inches, and the middle diameter, C D, 36 inches; what is the solidity of the spindle?

First $(Ae^2+Ce^2) \div CD = (24^2 + 18^2) \div 36 = (576 + 324) \div 36 = 900 \div 36 = 25$, the radius OC.

Second, $OC - Ce = 25 - 18 = 7$, $Oe$, the central distance.

Third, (by Problem XI., rule 5, page 104,) $Ce \div 2\,OC = 18 \div 50 = .36$, the tabular versed sine, against which stands .254550 the tabular segment.

Here $.254550 \times 50^2 = .254550 \times 2500 = 636.375$, the area of the segment A B C A.

Fourth, $\frac{1}{3} \mathrm{A}e^3 - (\frac{1}{2} \mathrm{ABCA} \times \mathrm{O}e) \times 12.5664 = [(24^3 \div 3) - (636.375 \div 2) \times 7] \times 12.5664 = [(13824 \div 3) - (318.1875 \times 7)] \times 12.5664 = (4608 - 2227.3125) \times 12.5664 = 2380.6875 \times 12.5664 = 29916.6714$, solidity of the spindle.

2. If the length of a circular spindle be 40 inches, and its middle diameter 30 inches, what is its solidity?

*Ans.* 17312.8886 inches.

## PROBLEM XXXI.

To find the solidity of the middle frustum of a circular spindle, its length, the middle diameter, and that of either of the ends, being given.

RULE 1.—Divide the square of half the length of the frustum by half the difference of the middle diameter, and that of either of the two ends; and half this quotient, added to one-half of the said difference, will give the radius of the circle.

2. Find the central distance, and the revolving area, as in the last problem.

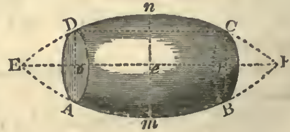
3. From the square of the radius take the square of the central distance, and the square root of the remainder will give half the length of the spindle.

4. From the square of half the length of the spindle take one-third of the square of half the length of the frustum, and multiply the remainder by the said half length.

5. From this product take that of the generating area and central distance, and the remainder multiplied by 6.2832 will give the solidity of the frustum.

### EXAMPLES.

1. What is the solidity of the middle frustum, A B C D, of a circular spindle, whose middle diameter, $nm$, is 36 inches, the diameter D A or C B, of the end 16 inches, and its length *or* 40 inches?



First, $\frac{1}{2}[Oe^2 \div (ne - Do)] + \frac{1}{2}(ne - Do) = \frac{1}{2}[20^2 \div (18 - 8)] + \frac{1}{2}(18 - 8) = \frac{1}{2}(400 \div 10) + (10 \div 2) = 20 + 5 = 25$, the radius of the circle.

Second, $25 - ne = 25 - 18 = 7$, the central distance.

And $ne - Do = 18 - 8 = 10$, the versed sine of the arc D$n$.

Then, by Problem XI., rule 5, page 104, $10 \div (25 \times 2) = 10 \div 50 = .2$, the tabular versed sine; against which stands .111823 the tabular segment.

Hence $.111823 \times 50^2 = .111823 \times 2500 = 279.5575$, the area of the revolving segment DC$n$D.

Again, by Problem 1, page 61, $or \times Do = 40 \times 8 = 320$, the area of the rectangle D$or$CD.

And DC$n$D + D$or$ CD $= 279.5575 + 320 = 599.5575$, the generating area O$r$C$n$D$o$.

Third, $\sqrt{(25^2 - 7^2)} = \sqrt{(625 - 49)} = \sqrt{576} = 24 =$ E$e$, half the length of the spindle.

Fourth, $\{[(Ee^2 - \frac{1}{3}Oe^2) \times Oe] - (OrCnDo \times 7)\} \times 6.2832 = \{[(24^2 - 133\frac{1}{3}) \times 20] - (599.5575 \times 7)\} \times 6.2832 = [(442\frac{2}{3} \times 20) - 4196.9025] \times 6.2832 = (8853.3333\frac{1}{3} - 4196.9025) \times 6.2832 = 4656.4308\frac{1}{3} \times 6.2832 = 29257.2862$ cubic inches, the solidity of the middle frustum A$m$BC$n$DA required.

2. The middle diameter of the frustum of a circular spindle is 2 feet 8 inches, the diameter at the end is 2 feet, and the length 3 feet 4 inches; what is the solidity?

*Ans.* 27285.0882 inches.

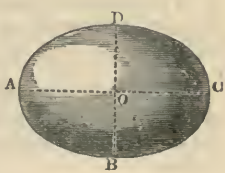
## PROBLEM XXXII.

To find the solidity of a spheroid, its two axes being given.

RULE.—Multiply the square of the revolving axis by the fixed axis, and this product again by .5236, or one-sixth of 3.1416, and it will give the solidity required.

### EXAMPLES.

1. In the prolate spheroid A B C D, the transverse or fixed axis, AC, is 3 ft, and the conjugate or revolving axis, D B, is 2 feet; what is the solidity?




Here $(2^2 \times 3) \times .5236 = (4 \times 3) \times .5236 = 12 \times .5236 = 6.2832$ feet, the solidity required.

2. What is the solidity of a prolate spheroid whose transverse or fixed axis is 4 feet 2 inches, and conjugate or revolving axis 3 feet 4 inches? *Ans.* 24.2407 ft.

3. What is the solidity of a prolate spheroid whose fixed axis is 8 feet 4 inches, and its revolving axis 5 feet? *Ans.* 109.0833 ft.

4. What is the solidity of an oblate spheroid whose conjugate or fixed axis is 5 feet, and its transverse or revolving axis 8 feet 4 inches? *Ans.* 181.8055 ft.

5. What is the solidity of an oblate spheroid whose fixed axis is 30 inches, and its revolving axis 40 inches? *Ans.* 14.5444 ft.

### PROBLEM XXXIII.

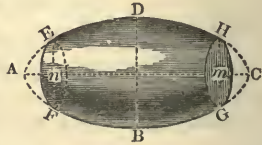
To find the solidity of the middle frustum of a spheroid, its length, the middle diameter, and that of either of the ends, being given.

CASE 1.—When the ends are circular, or perpendicular to the fixed axis.

RULE.—To twice the square of the middle diameter add the square of the diameter of either of the ends, and this sum multiplied by the length of the frustum, and the product again by .2618 (or one-twelfth of 3.1416), will give the solidity.

#### EXAMPLES.

1. In the middle frustum of a prolate spheroid E F G H, the middle diameter, B D, is 50 inches, and that of either of the ends E F or G H, is 40 inches, and its length, $nm$, 18 inches; what is its solidity?




Here $[(50^2\times2+40^2)\times18]\times.2618=[(2500\times2+1600)\times18]\times.2618=[(5000+1600)\times18]\times.2618=(6600\times18)\times.2618=118800\times.2618=31101.84$ cubic inches, the solidity required.

2. What is the solidity of the middle frustum of a prolate spheroid, the middle diameter being 5 feet, that of either of the two ends 3 feet, and the distance of the ends 6 feet 8 inches? *Ans.* 102.9746 feet.

3. What is the solidity of the middle frustum of an oblate spheroid, the middle diameter being 100 inches, that of either of the ends 80 inches, and the distance of the ends 36 inches? *Ans.* 248814.72 inches.

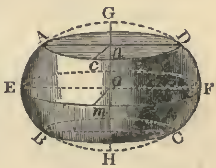
4. What is the solidity of the middle frustum of a prolate spheroid, the middle diameter being 5 feet, that of either of the ends 3 feet, and the distance of the ends 6 feet? *Ans.* 92.6772 ft.

CASE 2.—When the ends are elliptical or perpendicular to the revolving axis.

RULE 1.—Multiply twice the transverse diameter of the middle section by its conjugate diameter, and to this product add the product of the transverse and conjugate diameters of either of the ends.

2. Multiply the sum thus found, by the distance of the ends or the height of the frustum, and the product again by .2618, and it will give the solidity required.

EXAMPLES.




1. In the middle frustum, A B C D, of a prolate spheroid, the diameters of the middle section are 50 and 30 inches; those of the end 40 and 24 inches; and its height, 2 *on*, 18 inches; what is the solidity?

Here $\{[(50\times2\times30)+(40\times24)]\times18\}\times.2618=[(3000+960)\times18]\times.2618=(3960\times18)\times.2618=71280\times.2618=18661.104$ cubic inches, the solidity required.

2. In the middle frustum of a prolate spheroid, the diameters of the middle section are 100 and 60 inches; those of the end 80 and 48 inches; and the length 36 inches; what is the solidity? *Ans.* 86.3940 ft.

3. In the middle frustum of an oblate spheroid, the diameters of the middle section are 100 and 60 inches; those of the end 60 and 36 inches; and the length 80 inches; what is the solidity of the frustum? *Ans.* 171.6244 ft.

# A TABLE OF THE AREAS OF THE SEGMENTS OF A CIRCLE,

Whose diameter is Unity, and supposed to be divided into 1000 equal Parts.

| Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. |
|---|---|---|---|---|---|
| .001 | .000042 | .034 | .008273 | .067 | .022652 |
| .002 | .000119 | .035 | .008638 | .068 | .023154 |
| .003 | .000219 | .036 | .009008 | .069 | .023659 |
| .004 | .000337 | .037 | .009383 | .070 | .024168 |
| .005 | .000470 | .038 | .009763 | .071 | .024680 |
| .006 | .000618 | .039 | .010148 | .072 | .025195 |
| .007 | .000779 | .040 | .010537 | .073 | .025714 |
| .008 | .000951 | .041 | .010931 | .074 | .026236 |
| .009 | .001135 | .042 | .011330 | .075 | .026761 |
| .010 | .001329 | .043 | .011734 | .076 | .027289 |
| .011 | .001533 | .044 | .012142 | .077 | .027821 |
| .012 | .001746 | .045 | .012554 | .078 | .028356 |
| .013 | .001968 | .046 | .012971 | .079 | .028894 |
| .014 | .002199 | .047 | .013392 | .080 | .029435 |
| .015 | .002438 | .048 | .013818 | .081 | .029979 |
| .016 | .002685 | .049 | .014247 | .082 | .030526 |
| .017 | .002940 | .050 | .014681 | .083 | .031076 |
| .018 | .003202 | .051 | .015119 | .084 | .031629 |
| .019 | .003471 | .052 | .015561 | .085 | .032186 |
| .020 | .003748 | .053 | .016007 | .086 | .032745 |
| .021 | .004031 | .054 | .016457 | .087 | .033307 |
| .022 | .004322 | .055 | .016911 | .088 | .033872 |
| .023 | .004618 | .056 | .017369 | .089 | .034441 |
| .024 | .004921 | .057 | .017831 | .090 | .035011 |
| .025 | .005230 | .058 | .018296 | .091 | .035585 |
| .026 | .005546 | .059 | .018766 | .092 | .036162 |
| .027 | .005867 | .060 | .019239 | .093 | .036741 |
| .028 | .006194 | .061 | .019716 | .094 | .037323 |
| .029 | .006527 | .062 | .020196 | .095 | .037909 |
| .030 | .006865 | .063 | .020680 | .096 | .038496 |
| .031 | .007209 | .064 | .021168 | .097 | .039087 |
| .032 | .007558 | .065 | .021659 | .098 | .039680 |
| .033 | .007913 | .066 | .022154 | .099 | .040276 |

| Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. |
|---|---|---|---|---|---|
| .100 | .040875 | .141 | .067528 | .182 | .097674 |
| .101 | .041476 | .142 | .068225 | .183 | .098447 |
| .102 | .042080 | .143 | .068924 | .184 | .099221 |
| .103 | .042687 | .144 | .069625 | .185 | .099997 |
| .104 | .043296 | .145 | .070328 | .186 | .100774 |
| .105 | .043908 | .146 | .071033 | .187 | .101553 |
| .106 | .044522 | .147 | .071741 | .188 | .102334 |
| .107 | .045139 | .148 | .072450 | .189 | .103116 |
| .108 | .045759 | .149 | .073161 | .190 | .103900 |
| .109 | .046381 | .150 | .073874 | .191 | .104685 |
| .110 | .047005 | .151 | .074589 | .192 | .105472 |
| .111 | .047632 | .152 | .075306 | .193 | .106261 |
| .112 | .048262 | .153 | .076026 | .194 | .107051 |
| .113 | .048894 | .154 | .076747 | .195 | .107842 |
| .114 | .049528 | .155 | .077469 | .196 | .108636 |
| .115 | .050165 | .156 | .078194 | .197 | .109430 |
| .116 | .050804 | .157 | .078921 | .198 | .110226 |
| .117 | .051446 | .158 | .079649 | .199 | .111024 |
| .118 | .052090 | .159 | .080380 | .200 | .111823 |
| .119 | .052736 | .160 | .081112 | .201 | .112624 |
| .120 | .053385 | .161 | .081846 | .202 | .113426 |
| .121 | .054036 | .162 | .082582 | .203 | .114230 |
| .122 | .054689 | .163 | .083320 | .204 | .115035 |
| .123 | .055345 | .164 | .084059 | .205 | .115842 |
| .124 | .056003 | .165 | .084801 | .206 | .116650 |
| .125 | .056663 | .166 | .085544 | .207 | .117460 |
| .126 | .057326 | .167 | .086289 | .208 | .118271 |
| .127 | .057991 | .168 | .087036 | .209 | .119083 |
| .128 | .058658 | .169 | .087785 | .210 | .119897 |
| .129 | .059327 | .170 | .088535 | .211 | .120712 |
| .130 | .059999 | .171 | .089287 | .212 | .121529 |
| .131 | .060672 | .172 | .090041 | .213 | .122347 |
| .132 | .061348 | .173 | .090797 | .214 | .123167 |
| .133 | .062026 | .174 | .091554 | .215 | .123988 |
| .134 | .062707 | .175 | .092313 | .216 | .124810 |
| .135 | .063389 | .176 | .093074 | .217 | .125634 |
| .136 | .064074 | .177 | .093836 | .218 | .126459 |
| .137 | .064760 | .178 | .094601 | .219 | .127285 |
| .138 | .065449 | .179 | .095366 | .220 | .128113 |
| .139 | .066140 | .180 | .096134 | .221 | .128942 |
| .140 | .066833 | .181 | .096903 | .222 | .129773 |

| Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. |
|---|---|---|---|---|---|
| .223 | .130605 | .264 | .165780 | .305 | .202761 |
| .224 | .131438 | .265 | .166663 | .306 | .203683 |
| .225 | .132272 | .266 | .167546 | .307 | .204605 |
| .226 | .133108 | .267 | .168430 | .308 | .205527 |
| .227 | .133945 | .268 | .169315 | .309 | .206451 |
| .228 | .134784 | .269 | .170202 | .310 | .207376 |
| .229 | .135624 | .270 | .171089 | .311 | .208301 |
| .230 | .136465 | .271 | .171978 | .312 | .209227 |
| .231 | .137307 | .272 | .172867 | .313 | .210154 |
| .232 | .138150 | .273 | .173758 | .314 | .211082 |
| .233 | .138995 | .274 | .174649 | .315 | .212011 |
| .234 | .139841 | .275 | .175542 | .316 | .212940 |
| .235 | .140688 | .276 | .176435 | .317 | .213871 |
| .236 | .141537 | .277 | .177330 | .318 | .214802 |
| .237 | .142387 | .278 | .178225 | .319 | .215733 |
| .238 | ·143238 | .279 | .179122 | .320 | .216666 |
| .239 | .144091 | .280 | .180019 | .321 | .217599 |
| .240 | .144944 | .281 | .180918 | .322 | .218533 |
| .241 | .145799 | .282 | .181817 | .323 | .219468 |
| .242 | .146655 | .283 | .182718 | .324 | .220404 |
| .243 | .147512 | .284 | .183619 | .325 | .221340 |
| .244 | .148371 | .285 | .184521 | .326 | .222277 |
| .245 | .149230 | .286 | .185425 | .327 | .223215 |
| .246 | .150091 | .287 | .186329 | .328 | .224154 |
| .247 | .150953 | .288 | .187234 | .329 | .225093 |
| .248 | .151816 | .289 | .188140 | .330 | .226033 |
| .249 | .152680 | .290 | .189047 | .331 | .226974 |
| .250 | .153546 | .291 | .189955 | .332 | .227915 |
| .251 | .154412 | .292 | .190864 | .333 | .228858 |
| .252 | .155280 | .293 | .191775 | .334 | .229801 |
| .253 | .156149 | .294 | .192684 | .335 | .230745 |
| .254 | .157019 | .295 | .193596 | .336 | .231689 |
| .255 | .157890 | .296 | .194509 | .337 | .232634 |
| .256 | .158762 | .297 | .195422 | .338 | .233580 |
| .257 | .159636 | .298 | .196337 | .339 | .234526 |
| .258 | .160510 | .299 | .197252 | .340 | .235473 |
| .259 | .161386 | .300 | .198168 | .341 | .236421 |
| .260 | .162263 | .301 | .199085 | .342 | .237369 |
| .261 | .163140 | .302 | .200003 | .343 | .238318 |
| .262 | .164019 | .303 | .200922 | .344 | .239268 |
| .263 | .164899 | .304 | .201841 | .345 | .240218 |

| Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. |
|---|---|---|---|---|---|
| .346 | .241169 | .387 | .280668 | .428 | .320948 |
| .347 | .242121 | .388 | .281642 | .429 | .321938 |
| .348 | .243074 | .389 | .282617 | .430 | .322928 |
| .349 | .244026 | .390 | .283592 | .431 | .323918 |
| .350 | .244980 | .391 | .284568 | .432 | .324909 |
| .351 | .245934 | .392 | .285544 | .433 | .325900 |
| .352 | .246889 | .393 | .286521 | .434 | .326892 |
| .353 | .247845 | .394 | .287498 | .435 | .327882 |
| .354 | .248801 | .395 | .288476 | .436 | .328874 |
| .355 | .249757 | .396 | .289453 | .437 | .329866 |
| .356 | .250715 | .397 | .290432 | .438 | .330858 |
| .357 | .251673 | .398 | .291411 | .439 | .331850 |
| .358 | .252631 | .399 | .292390 | .440 | .332843 |
| .359 | .253590 | .400 | .293369 | .441 | .333836 |
| .360 | .254550 | .401 | .294349 | .442 | .334829 |
| .361 | .255510 | .402 | .295330 | .443 | .335822 |
| .362 | .256471 | .403 | .296311 | .444 | .336816 |
| .363 | .257433 | .404 | .297292 | .445 | .337810 |
| .364 | .258395 | .405 | .298273 | .446 | .338804 |
| .365 | .259357 | .406 | .299255 | .447 | .339798 |
| .366 | .260320 | .407 | .300238 | .448 | .340793 |
| .367 | .261284 | .408 | .301220 | .449 | .341787 |
| .368 | .262248 | .409 | .302203 | .450 | .342782 |
| .369 | .263213 | .410 | .303187 | .451 | .343777 |
| .370 | .264178 | .411 | .304171 | .452 | .344772 |
| .371 | .265144 | .412 | .305155 | .453 | .345768 |
| .372 | .266111 | .413 | .306140 | .454 | .346764 |
| .373 | .267078 | .414 | .307125 | .455 | .347759 |
| .374 | .268045 | .415 | .308110 | .456 | .348755 |
| .375 | .269013 | .416 | .309095 | .457 | .349752 |
| .376 | .269982 | .417 | .310081 | .458 | .350748 |
| .377 | .270951 | .418 | .311068 | .459 | .351745 |
| .378 | .271920 | .419 | .312054 | .460 | .352741 |
| .379 | .272890 | .420 | .313041 | .461 | .353739 |
| .380 | .273861 | .421 | .314029 | .462 | .354736 |
| .381 | .274832 | .422 | .315016 | .463 | .355732 |
| 382 | .275803 | .423 | .316004 | .464 | .356730 |
| .383 | .276775 | .424 | .316992 | .465 | .357727 |
| .384 | .277748 | .425 | .317981 | .466 | .358725 |
| .385 | .278721 | .426 | .318970 | .467 | .359723 |
| .386 | .279694 | .427 | .319959 | .468 | .360721 |

| Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. | Versed Sine. | Seg. Area. |
|---|---|---|---|---|---|
| .469 | .361719 | .480 | .372764 | .491 | .383699 |
| .470 | .362717 | .481 | .373703 | .492 | 384699 |
| .471 | .363715 | .482 | .374702 | .493 | .385699 |
| .472 | .364713 | .483 | .375702 | .494 | .386699 |
| .473 | .365712 | .484 | .376702 | .495 | .387699 |
| .474 | .366710 | .485 | .377701 | .496 | .388699 |
| .475 | .367709 | .486 | .378701 | .497 | .389699 |
| .476 | .368708 | .487 | .379700 | .498 | .390699 |
| .477 | .369707 | .488 | .380700 | .499 | .391699 |
| .478 | .370706 | .489 | .381699 | .500 | .392699 |
| .479 | .371705 | .490 | .382699 | | |

THE END.